

## Appendix

### Listing of the FORTRAN-program used to draw the Boy surface and its deformations

by *Raymond Ripp*

The program was run on a PS 300 Evans-Sutherland connected to a VAX computer. It generates the Boy surface, the generalizations with an axis of  $n$ -fold symmetry, and all the steps of the deformation giving the Roman surface.

#### *Parameters:*

$G$  is the parameter of *deformation*

$n$  is the parameter of *symmetry*

$g = 0$	$n = 3$	Plates 21, 22, 56, 57
$g = 1$	$n = 3$	Plates 39, 40, 41, 42, 43
$g = 1/\sqrt{3}$	$n = 3$	Plates 51, 52
$g = 0, 4$	$n = 3$	Plate 53
$g = (\sqrt{2}-1)^2$	$n = 3$	Plate 54
$g = 1/1000$	$n = 3$	Plate 55
$g = 1$	$n = 2$	Cover
$g = 0$	$n = 4$	Plates 59, 60, 61
$g = 0$	$n = 5$	Plates 62, 63
$g = 1$	$n = 5$	Plate 64

```

program boy
call parametres
call surface
call intersection

call pssexit
stop
end

```

```
subroutine parametres
```

```
include 'boycom.inc'
```

```
ne = questionR('$how many ellipses do you want to draw :@')
np = questionR('$how many points in each ellipse ..... :@')
g = questionR('$give the value of g (0 to 1) ..... :@')
n = questionI('$give the value of n (n>=2 sym.order) :@')
npi= questionR('$how many points in the intersection . :@')
```

```
return
end
```

```
subroutine surface
```

```
include 'boycom.inc'
```

```
logical pl
```

```
pi=3.1415927
r2=sqrt(2.)
```

```
call newgraph
```

```
do i=0,2*ne-1          ! for each ellipse
  e=float(i)
  pl=.false.
  do j=0,np           ! for each point of the ellipse
    h=float(j)

    a=h*pi/np-pi/2
    b=e*pi/ne

    c=cos(a)/(1-(g*sin(2*a)*sin(n*b))/r2)

    x=c*((r2/n)*cos(a)*cos((n-1)*b)+(n-1)*sin(a)*cos(b)/n)
    y=c*((r2/n)*cos(a)*sin((n-1)*b)-(n-1)*sin(a)*sin(b)/n)
    z=c*cos(a)-2./3.

    call graph(pl,x,y,z)
    pl=.true.
  end do
end do
```

```
end do
```

```
call endgraph
```

```
return
end
```

```

subroutine intersection
include 'boycom.inc'
s2=sqrt(2.)
if (n.eq.2)                call interNeq2
if (n.eq.3.and.g.eq.0)    call interNeq3Geq
if (n.eq.3.and.g.gt.0.and.g.lt.(s2-1)**2) call interNeq3Glt
if (n.eq.3.and.g.gt.0.and.g.ge.(s2-1)**2) call interNeq3Gge
return
end

subroutine interNeq2      ! intersection N=2
include 'boycom.inc'
logical pl
call newgraph
pl=.false.
do i=0,npi
    ri=float(i)
    t=-0.5*atan(4./3.) + (ri/(2*npi))*(atan(2.)+atan(4./3.))
    c2t=cos(2*t)
    s2t=sin(2*t)
    r=5*(3*c2t-4*s2t)* sqrt( abs((3*s2t+4*c2t)*(2*c2t-s2t)) )
1    x0= 3.*s2t*s2t*s2t -288.*s2t*s2t*c2t
    +91.*s2t*c2t*c2t - 54.*c2t*c2t*c2t
    x1=r*cos(t)
    x2=r*sin(t)
    x3=25*(s2t-2*c2t)
    x=x1/x0
    y=x2/x0
    z=x3/x0-2./3.
    call graph(pl,X,Y,Z)
    pl=.true.
end do

```

```

pl=.false.
do i=0,npi
  ri=float(i)

  t=-0.5*atan(4./3.) + (ri/(2*npi))*(atan(2.)+atan(4./3.))

  c2t=cos(2*t)
  s2t=sin(2*t)

  r=5*(3*c2t-4*s2t)* sqrt( abs((3*s2t+4*c2t)*(2*c2t-s2t)) )

1  x0= 3.*s2t*s2t*s2t -288.*s2t*s2t*c2t
   +91.*s2t*c2t*c2t - 54.*c2t*c2t*c2t
  x1=r*cos(t)
  x2=r*sin(t)
  x3=25*(s2t-2*c2t)

  x=-x2/x0
  y= x1/x0
  z= x3/x0-2./3.

  call graph(pl,X,Y,Z)
  pl=.true.
end do

pl=.false.
do i=0,npi

  ri=float(i)

  t=-0.5*atan(4./3.) + (ri/(2*npi))*(atan(2.)+atan(4./3.))

  c2t=cos(2*t)
  s2t=sin(2*t)

  r=5*(3*c2t-4*s2t)* sqrt( abs((3*s2t+4*c2t)*(2*c2t-s2t)) )

1  x0= 3.*s2t*s2t*s2t -288.*s2t*s2t*c2t
   +91.*s2t*c2t*c2t - 54.*c2t*c2t*c2t
  x1=r*cos(t)
  x2=r*sin(t)
  x3=25*(s2t-2*c2t)

  x= x2/x0
  y=-x1/x0
  z= x3/x0-2./3.

  call graph(pl,X,Y,Z)
  pl=.true.
end do

```

```

pl=.false.
do i=0,npi
  ri=float(i)

  t=-0.5*atan(4./3.) + (ri/(2*npi))*(atan(2.)+atan(4./3.))

  c2t=cos(2*t)
  s2t=sin(2*t)

  r=5*(3*c2t-4*s2t)* sqrt( abs((3*s2t+4*c2t)*(2*c2t-s2t)) )

1  x0= 3.*s2t*s2t*s2t -288.*s2t*s2t*c2t
   +91.*s2t*c2t*c2t - 54.*c2t*c2t*c2t
  x1=r*cos(t)
  x2=r*sin(t)
  x3=25*(s2t-2*c2t)

  x=-x1/x0
  y=-x2/x0
  z= x3/x0-2./3.

  call graph(pl,X,Y,Z)
  pl=.true.
end do

call endgraph

return
end

```

```

subroutine interNeq3Geq ! intersection N=3 G=0

```

```

include 'boycom.inc'

```

```

R2=SQRT(2.)

```

```

R3=SQRT(3.)

```

```

call newgraph

```

```

C line 1

```

```

X1=20.*R2/3.

```

```

X2=0.

```

```

X3=-6.-2./3.

```

```

call graph(.false.,X1,X2,X3)

```

```

X1=-20.*R2/3.

```

```

X2=0.

```

```

X3=22./3.-2./3.

```

```

call graph(.true.,X1,X2,X3)

```

```

C line 2

```

```

X1=-10.*R2/3.

```

```

X2= 10.*R2/R3

```

```

X3=-6.-2./3.

```

```

call graph(.false.,X1,X2,X3)

```

```

X1=10*R2/3.
X2=-10*R2/R3
X3=22./3.-2./3.
call graph(.true.,X1,X2,X3)

```

C line 3

```

X1=-10.*R2/3.
X2=-10.*R2/R3
X3=-6.-2./3.
call graph(.false.,X1,X2,X3)

```

```

X1=10.*R2/3.
X2=10.*R2/R3
X3=22./3.-2./3.
call graph(.true.,X1,X2,X3)

```

```

pl=.false.
do i=1,npi-1
    t=-pi/6. + pi/3.*float(i)/npi
    uscos=1/(3*cos(3*t))
    x=uscos*2*s2*cos(t)
    y=uscos*2*s2*sin(t)
    z=-2./3.
    call graph(pl,x,y,z)
    pl=.true.
end do

```

```

pl=.false.
do i=1,npi-1
    t=pi/6. + pi/3.*float(i)/npi
    uscos=1/(3*cos(3*t))
    x=uscos*2*s2*cos(t)
    y=uscos*2*s2*sin(t)
    z=-2./3.

```

```

    call graph(pl,x,y,z)
    pl=.true.

```

end do

```

pl=.false.
do i=1,npi-1
    t=pi/2. + pi/3.*float(i)/npi
    uscos=1/(3*cos(3*t))
    x=uscos*2*s2*cos(t)
    y=uscos*2*s2*sin(t)
    z=-2./3.
    call graph(pl,x,y,z)
    pl=.true.

```

end do

call endgraph

```

return
end

```

```
subroutine interNeq3Gge      ! intersection n=3 g>(sqrt(2)-1)**2
include 'boycom.inc'

logical pl

pi=3.1415927
r2=sqrt(2.)

call newgraph

a=atan(g)

pl=.false.
do i=0,npi
    t=(pi*float(i))/npi
    apt=a+t

    ss=3*(3*sin(2*a)+sin(6*apt))
    if ( abs(ss).le.0.01 ) then
        x=0.
        y=0.
        z=0.
        goto 9
    end if

    r=(4*r2*cos(a)*sin(3*apt))/ss

    x=r*cos(t)
    y=r*sin(t)
    z=6*sin(2*a)/ss-2./3.

    call graph(pl,x,y,z)
    pl=.true.
end do

call endgraph

return
end
```

```

subroutine interNeq3Glt          ! intersection n=3 0<g<(sqrt(2)-1)**:
include 'boycom.inc'

logical pl
real      aa(5000),bb(5000),cc(5000)

pi=3.1415927
r2=sqrt(2.)
r3=sqrt(3.)

call newgraph

a=atan(g)

t1=-asin(3*sin(2*a))/6.-a
t2= asin(3*sin(2*a))/6.-a+pi/6.
t3=-asin(3*sin(2*a))/6.-a+pi/3.
t4=t1
t5=t2

do ir=1,2

    do ih=1,npi+1
        h=float(ih)
        xh=-1+2*(h)/(npi+2.)
        t=t4+(t5-t4)*(0.5+xh/(1.+xh*xh))
        u=3*(3*sin(2*a)+sin(6*(a+t)))
        v=4*r2*cos(a)*cos(t)*sin(3*(a+t))
        w=4*r2*cos(a)*sin(t)*sin(3*(a+t))
        aa(ih)=v/u
        bb(ih)=w/u
        cc(ih)=6.*sin(2*a)/u
    end do

    xx1=-20.*r2/3.
    yy1=0.
    xx2=-10.*r2/3.
    yy2=-10.*r2/r3

    do is=0,2
        pl=.false.
        if (ir.eq.1.and.g.le.0.01) then
            x=xx1
            y=yy1
            z=22./3.-2./3.
            call graph(.false.,x,y,z)
            pl=.true.
        end if
        do ih=1,npi
            x=aa(ih)
            y=bb(ih)
            z=cc(ih)-2./3.
            call graph(pl,x,y,z)
            pl=.true.
        end do
        if (ir.eq.2.and.g.le.0.01) then
            x=xx2
            y=yy2
            z=-6.-2./3.
        end if
    end do
end do

```



```

                call graph(.true.,x,y,z)
            end if

            if (ir.eq.1.and.g.le.0.01) then
                x=-(xx1+r3*yy1)/2.
                y=(-yy1+r3*xx1)/2.
                xx1=x
                yy1=y
            end if

            if (ir.eq.2.and.g.le.0.01) then
                x=-(xx2+r3*yy2)/2.
                y=(-yy2+r3*xx2)/2.
                xx2=x
                yy2=y
            end if

            do ih=1,npi+1
                x=-(aa(ih)+r3*bb(ih))/2.
                y=(-bb(ih)+r3*aa(ih))/2.
                aa(ih)=x
                bb(ih)=y
            end do

        end do

        t4=t2
        t5=t3
    end do

    call endgraph

    return
end

```

```

subroutine newgraph
include 'amoi:psamoi.inc'
character*2 cnum
    num=num+1
    write(cnum,'(i2.2)')num
    buf='boyobj'//cnum//':='vec item '
    call pssbuf

```

```

return
end

```

```

subroutine endgraph
include 'amoi:psamoi.inc'
    call pssvsto(0)

```

```

return
end

```

```

subroutine graph(pl,x,y,z)
logical pl
include 'amoi:psamoi.inc'
    call pssvsto(4,pl,x,y,z)
return
end

```

## Bibliography

- [AP] Apéry, F.  
La surface de Boy, *Adv. in Math.*, Vol 61, No 3, Sept 1986.
- [AR] Arnold, V. I.; Gussein-Zade, S. M.; Varchenko, A.  
Singularities of differentiable maps, Vol. 1, *Monographs in Mathematics*, Vol. 82, Birkhauser, 1985.
- [BA] Banchoff, T.  
Triple points and surgery of immersed surfaces, *Proc. Amer. Math. Soc.* vol46 n3, p. 407–413, 1974.
- [BM] Banchoff, T.; Max, N.  
Every sphere eversion has a quadruple point, *Contributions to analysis and geometry* (Baltimore, Md., 1980), p. 191–209, Johns Hopkins Univ. Press, Baltimore, Md., 1981.
- [BON] Bonahon, F.  
Cobordism of automorphisms of surfaces, *Ann. Sc. Ec. Norm. Sup.*, 4e série, t. 16, p. 237–270, 1983.
- [BOU] Bourbaki, N.  
*Topologie générale*, Hermann, Paris, 1971.
- [BOY] Boy, W.  
über die Curvatura integra und die Topologie geschlossener Flächen, *Math. Ann.* 57, 151–184, 1903.
- [CE] Cerf, J.  
Sur les difféomorphismes de la sphère de dimension trois ( $\Gamma_4 = 0$ ), *Lect. Notes in Math.* 53, Springer, 1968.
- [DA] Darboux, G.  
*Théorie des surfaces*, Gauthiers-Villars, 1914.
- [DY] von Dyck, W.  
Beiträge zur Analysis situs I, *Math. Ann.*, t. 32, p. 457–512, Leipzig, 1888.
- [FI] Fischer, G.  
*Mathematical Models*, Vieweg, 1986.
- [FR] Frégier, M.  
Théorèmes nouveaux sur les lignes et surfaces du second ordre, *Annales de Gergonne*, VI n° VIII fév. 1816.
- [GO] Golubitsky, M.; Guillemin, V.  
*Stable Mappings and Their Singularities*, Springer-Verlag New York, 1973.

- [GRAM] Gramain, A.  
Rapport sur la théorie classique des noeuds (1ère partie), Séminaire Bourbaki, vol. 1975/76, Exposés 471–488), Lectures Notes in Math., 567, Springer, 1977.
- [GRAS] Grassmann, H.  
Die Ausdehnungslehre vollständig und in strenger Form, Verlag von Th. Enslin, Berlin, 1862.
- [GRI] Griffiths, H. B.  
Surfaces, Cambridge University Press, 1976.
- [HA] Haefliger, A.  
Quelques remarques sur les applications différentiables d'une surface dans le plan, Ann. Inst. Fourier, 10, p. 47–60, Grenoble, 1960.
- [HI] Hirsch, M. W.  
Differential Topology, Springer-Verlag, New York, 1976.
- [HO] Hopf, H.  
Differential geometry in the large, S. 104, Springer LNM 1000, 1983.
- [KL] Klein, F.  
Gesammelte Mathematische Abhandlungen, J. Springer, Berlin, 1923.
- [MAR] Martinet, J.  
Singularities of Smooths Functions and Maps, London Math. Soc., Lect. Note Ser. 58, Cambridge, 1982.
- [MAS] Massey, W. S.  
Algebraic Topology: An Introduction, Springer-Verlag, New York, 1967.
- [MO1] Morin, B.  
Formes canoniques des singularités d'une application différentiable, CRAS, t. 260, p. 5662–5665 et 6503–6506, Paris, 1965.
- [MO2] Morin, B.  
Equations du retournement de la sphère, CRAS série A, t. 287, 879–882, Paris, 1978.
- [MP] Morin, P.; Petit, J.-P.  
Le retournement de la sphère, Pour la Science, 15, p. 34–49, 1979.
- [PE] Petit, J.-P.; Souriau, J.  
Une représentation analytique de la surface de Boy, CRAS série I, t. 293, 269–272, 1981.
- [PI] Pinkall, U.  
Regular homotopy classes of immersed surfaces, to appear in Topology 1986.
- [PO] Pont, J.-C.  
La topologie algébrique des origines à Poincaré, P.U.F. Paris, 1974.
- [RE] Reinhardt, C.  
Zu Möbius' Polyedertheorie, Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Leipzig, 1885.

- [SC] Schilling, F.  
über die Abbildung der projektiven Ebene auf eine geschlossene singularitätenfreie Fläche im erreichbaren Gebiet des Raumes, *Math. Ann.* 92, 69–79, 1924.
- [SM] Smale, S.  
A classification of immersions of the two-sphere, *Transactions A.M.S.*, 90, p. 281–290, 1959.
- [SP] Spivak, M.  
A Comprehensive Introduction to Differential Geometry, vol. 1, Publish or Perish, Inc., Berkeley, 1979.
- [WA] Wallace Collao, M.  
Singularités de codimension deux des surfaces, Thèse de 3e cycle, Publication IRMA, Strasbourg, 1981.
- [WE] Weierstrass, K.  
Zwei spezielle Flächen vierter Ordnung, *Jacob Steiner's Gesammelte Werke* Bd. II, S. 741–742.
- [WH1] Whitney, H.  
On the topology of differentiable manifolds, *Lectures in topology*, Univ. Mich. press, p. 101–141, Ann Arbor, 1941.
- [WH2] Whitney, H.  
The General Type of Singularity of a Set of  $2n-1$  Smooth Functions of  $n$  Variables, *Duke Journal of Math.*, Ser. 2, 45, p. 220–293, 1944.

# Subject Index

- alternating 60
- annulus 2
- apparent contour 56
- atlas 42
  
- ball (closed unit) 1
- ball (open unit) 1
- base-point 22
- boundary 1
- Boy immersion 50
- Boy surface (combinatorial) 20
- Boy surface (direct) 51, 52, 79
- Boy surface (opposite) 51
  
- canonical map 2
- class 23
- class  $C^r$  43
- closed 1
- complex 5
- 0-complex 5
- 1-complex 5
- 2-complex 5
- confluence (elliptic) 71
- confluence (hyperbolic) 71
- confluence of Whitney umbrellas 70
- connected (pathwise) 22
- contingent 63
- covering (connected) 28
- covering of B of projection p and base B 26
- covering (n-sheeted) 27
- covering (trivial) 27
- critical point 45, 56
- critical set 56
- critical value 56
- cross-cap 40
- cross-cap (combinatorial) 18
- curve 1
- curve of umbrellas 70, 71
  
- degree 25
- degree (geometric) 106
- $C^r$ -diffeomorphism 43
- differentiable 43
- differentiable (r-times) 43
- discrete 27
- disk 2
- disk (open) 2
  
- edge 5
- $C^r$ -embedding 44
  
- embedding (topological) 1
- equivalent 22, 89
- Euler characteristic 5
- eversion of the sphere 103
- exact 29
  
- fiber above b 26
- fiber of the covering 26
- flag 14
- fold 57
- Frégier point 32
- fundamental group 23
  
- generic 57
- genus 13
- germ at a with value in 63
- graph 5
  
- half-tangent 91
- halfway model 103
- handkerchief folded into quarters 56
- handle of the Whitney umbrella 63
- handlebody of dimension 3 and genus 3 102
- Hessian quadratic form 45
- homogeneous of even degree 54
- homography 14
- homotopy 6
- homotopy of links 93
- homotopy (regular) 50
- homotopy type 6
- Hopf fibration 84
- hypocycloid (elongated) 77
  
- $C^r$ -immersion 48
- immersion (topological) 41
- index 45
- invariant under the antipodal action 54
- isotopic (ambient) 41
- isotopy ( $C^r$ -ambient) 51
  
- Jordan curve theorem 4
  
- Klein bottle 13
  
- lemniscate of Bernoulli 48, 85
- lift 57
- line 14, 31
- line (complex projective) 18
- line (projective) 14
- link 91

- linking number 92
- local coordinate system at  $x$  42
- loop 9
- loop based at 22
- loop (constant) 23
- loop (simple) 4
  
- manifold (n-dimensional) 1
- manifold (without boundary) 1
- manifold (smooth) 42
- maximum 45
- minimum 45
- Möbius strip 3, 14
- Morse function 45
  
- non-degenerate 45
- nondegenerate critical point of index  $r$  61
- nonorientable 4
  
- orientable 4, 88
- orientation-preserving 89
- oval 30
  
- pâquerette de Mélibée 85
- path 8
- path (opposite) 8
- path (piecewise smooth) 91
- path (simple) 9
- path (smooth) 91
- 1-pencil of conics 37
- 2-pencil of conics 36
- 3-pencil of conics 37
- permutation 28
- plane (projective) 8
- plane (real projective) 8
- pleat 57
- Plücker conoid 68
- pole of the Boy surface 74
- pole of the Roman surface 74
- projective group 14
- projective plane of Boy type (immersed) 90
- projective space (n-dimensional) 32
- projective space (n-dimensional real) 43
  
- quotient 2
- quotient topology 2
  
- rank 44
- rank theorem 61
- rational 79
- resultant 60
- retraction 93
- Rheinhardt heptahedron 17
- Riemannian metric 83
- Roman surface 37, 40
  
- saddle 45
- saturated 2
- segment (unit) 1
- self-intersection point 48
- self-intersection set 48
- self-transversal 84
- singular point 62
- singularity at a 62
- smooth 43
- smooth structure on  $M$  42
- smooth structure (oriented) 88
- smooth vector bundle of dimension 1 with base 94
- sphere (unit) 2
- stable 57
- stable at a 61
- starlike 23
- Steiner surface 36, 37
- strong deformation retract 93
- submanifold 86
- sum (connected) 11, 50
- sum (of paths) 9
- sum (topological) 1
- support 25, 91
- surface 1
- surface (embedded) 45
- surface (immersed closed) 50
- suspended umbrella 70
  
- tacnode 80
- tangent bundle 82, 83
- tangent bundle (unit) 83
- tangent space 82
- tangent space to  $M$  at a 83
- tangent vector to  $M$  at a 83
- three-bladed propeller 20
- three-bladed propeller (direct) 51, 90
- tied vector 82
- torus 2
- transversally 59
- trivial 93
- tubular neighborhood 93
- type  $D_1$  102
  
- umbilic (hyperbolic) 62
- unfold 69
  
- Veronese surface 37
- Veronese embedding 40
- vertex 5
  
- Whitney umbrella 62
- Whitehead link 93
- winding number 103

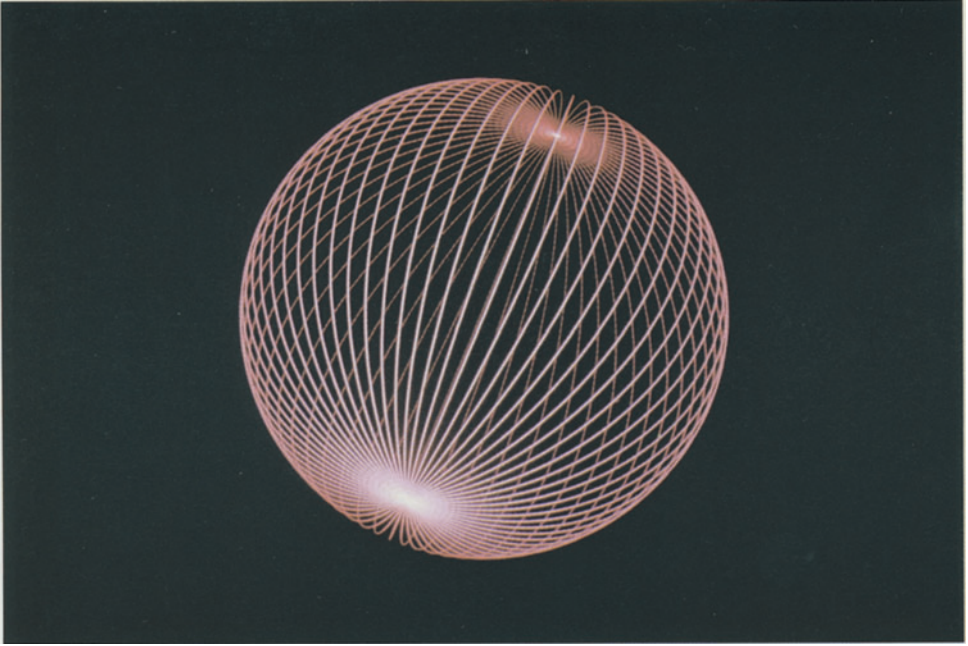
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## Plate Index

1	4, 13, 45	23	63, 68	45	79
2	57	24	68	46	52, 79
3	58	25	72	47	52, 79
4	58	26	72	48	79
5	59	27	72	49	79
6	59	28	72	50	79
7	3, 13, 44	29	72	51	81
8	44	30	72	52	81
9	50	31	55, 74	53	81
10	13, 44, 47	32	55, 74	54	81
11	11, 47	33	103	55	81
12	13, 44, 47	34	55	56	81
13	48	35	55	57	81
14	13	36	60	58	105
15	13, 49	37	60	59	104
16	13, 49	38	60	60	104
17	86	39	78, 104	61	104
18	41, 63	40	78, 79, 81, 104	62	104
19	41, 63	41	78, 79, 81, 104	63	104
20	41, 63	42	78, 90	64	104
21	37, 41, 63, 74	43	78, 90		
22	37, 41, 63, 74	44	79	Cover	104

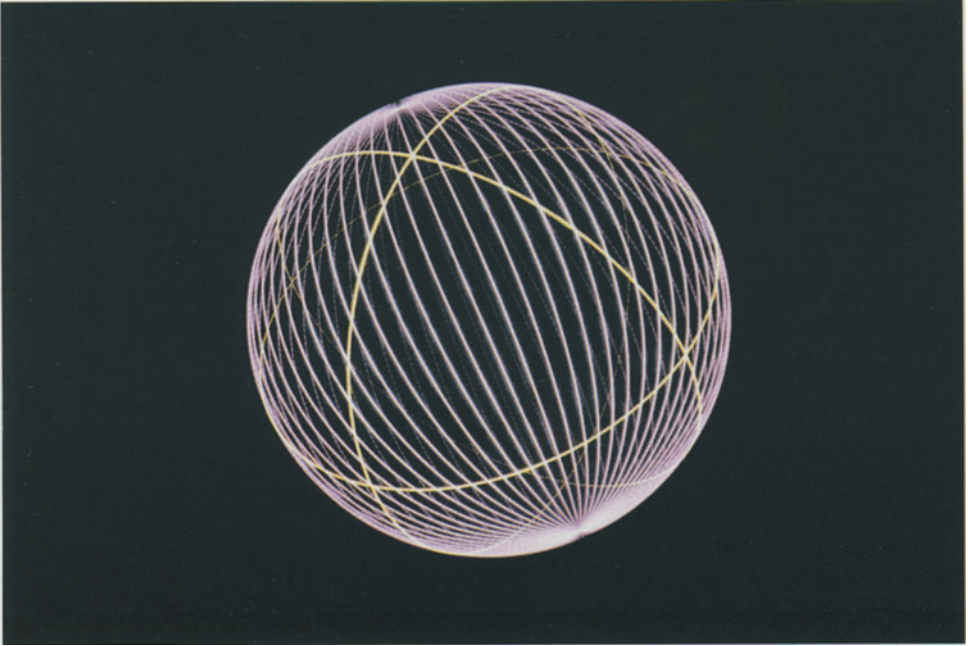
# Plates

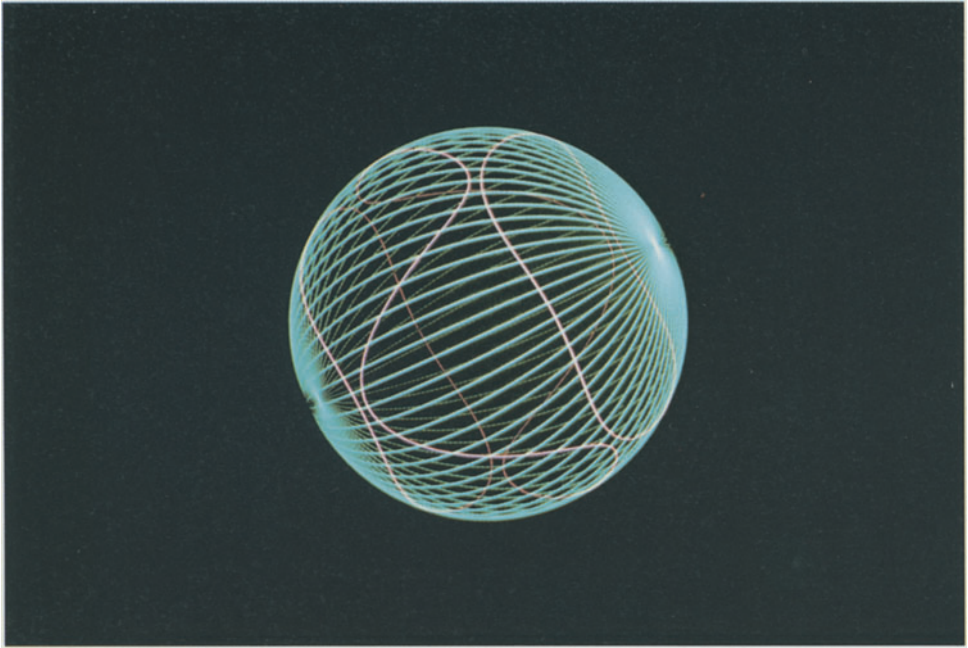




1 Sphere

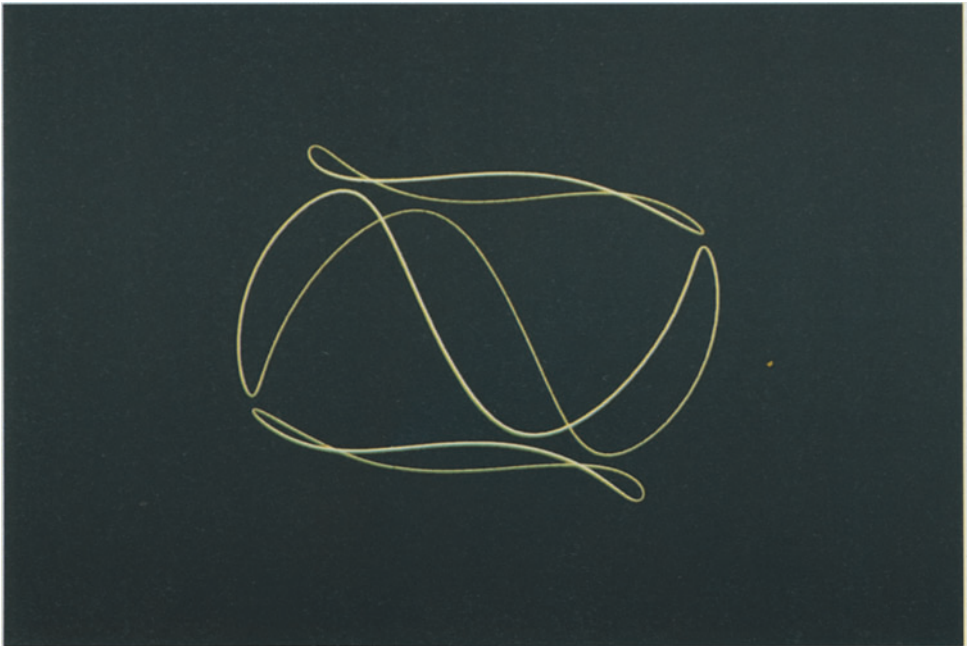
2 Sphere with three great circles

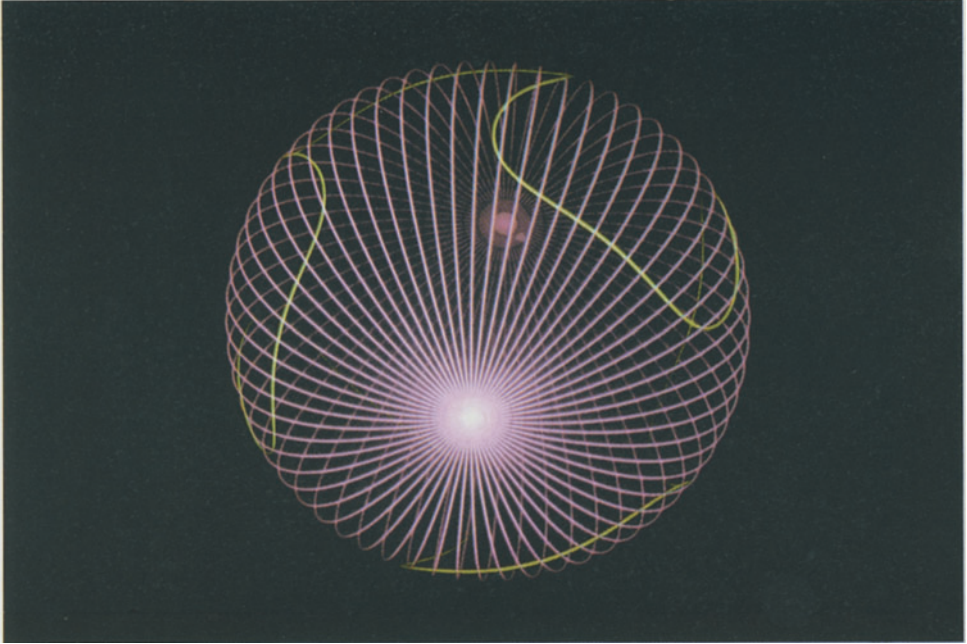




3 Sphere with a nonconnected perturbation of three great circles

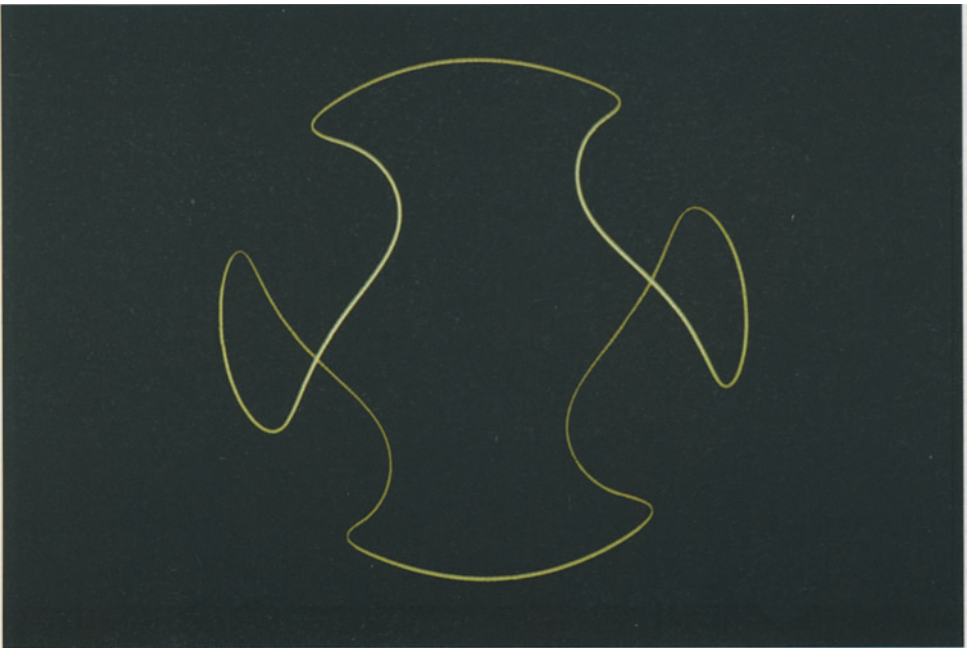
4 Nonconnected perturbation of three great circles



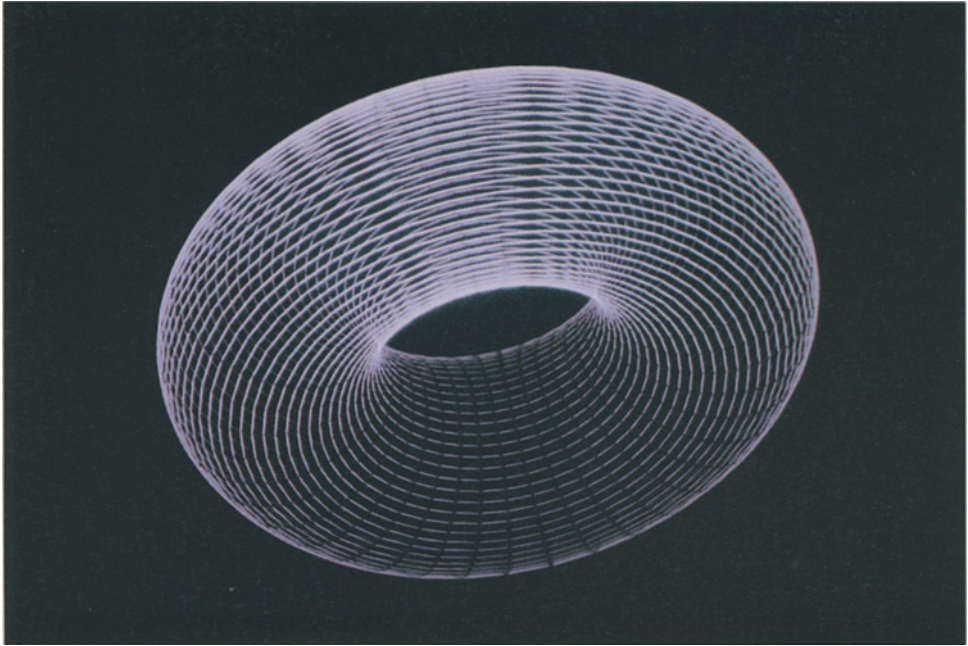


5 Sphere with a connected perturbation of three great circles

6 Connected perturbation of three great circles

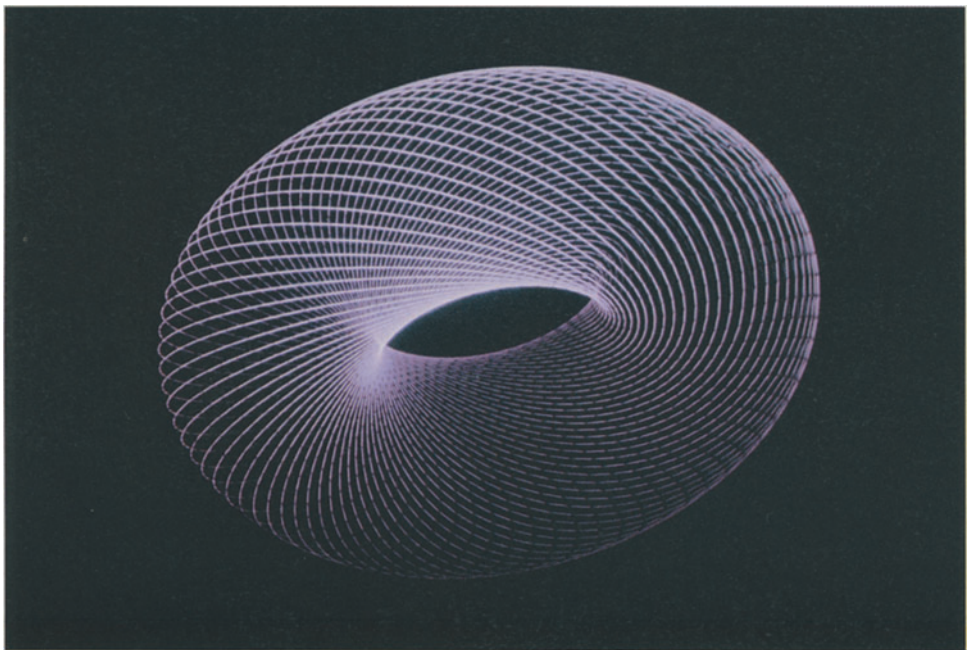


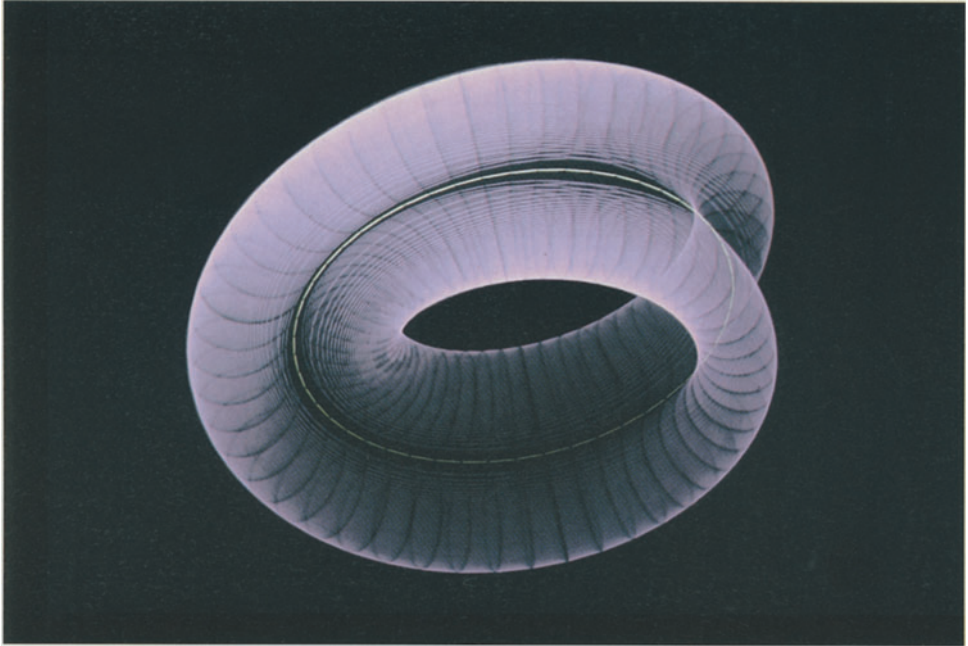




7 Standard torus

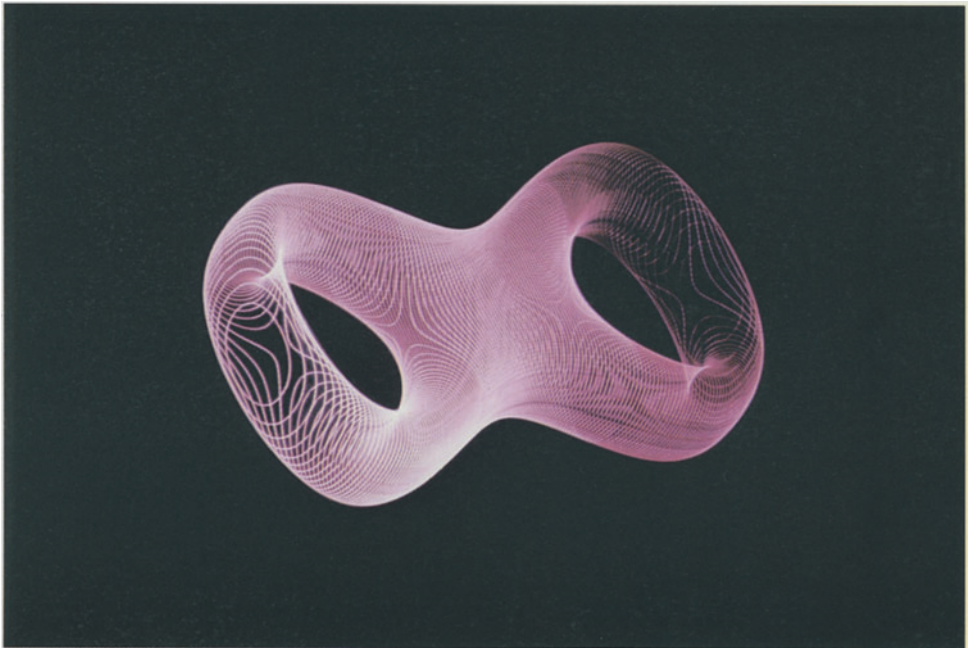
8 Torus generated by Villarceau circles

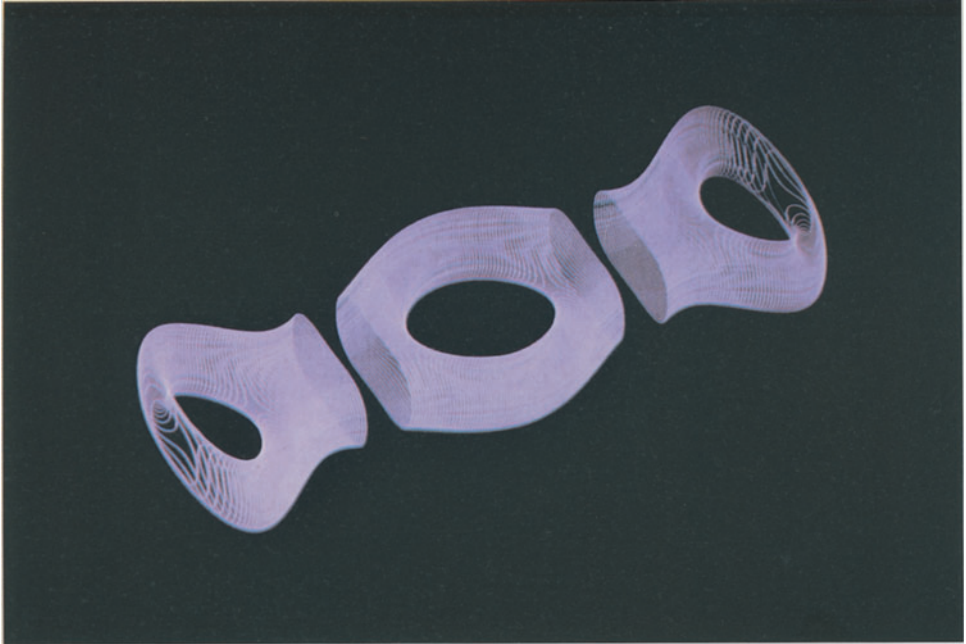




9 Immersed Klein bottle

10 Oriented closed surface of genus two



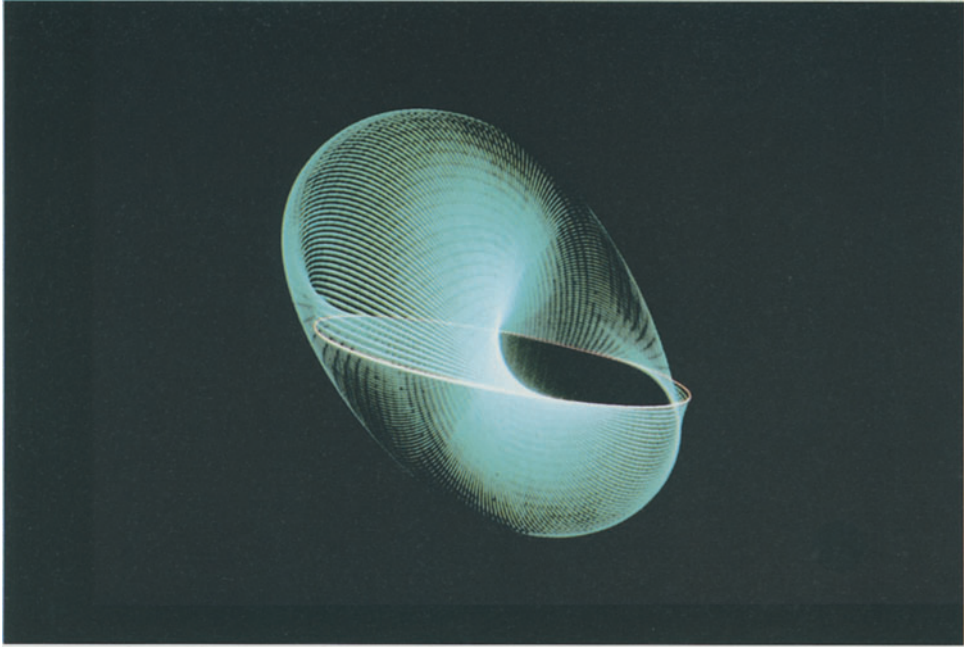


11 Connected sum of three tori

12 Oriented closed surface of genus three

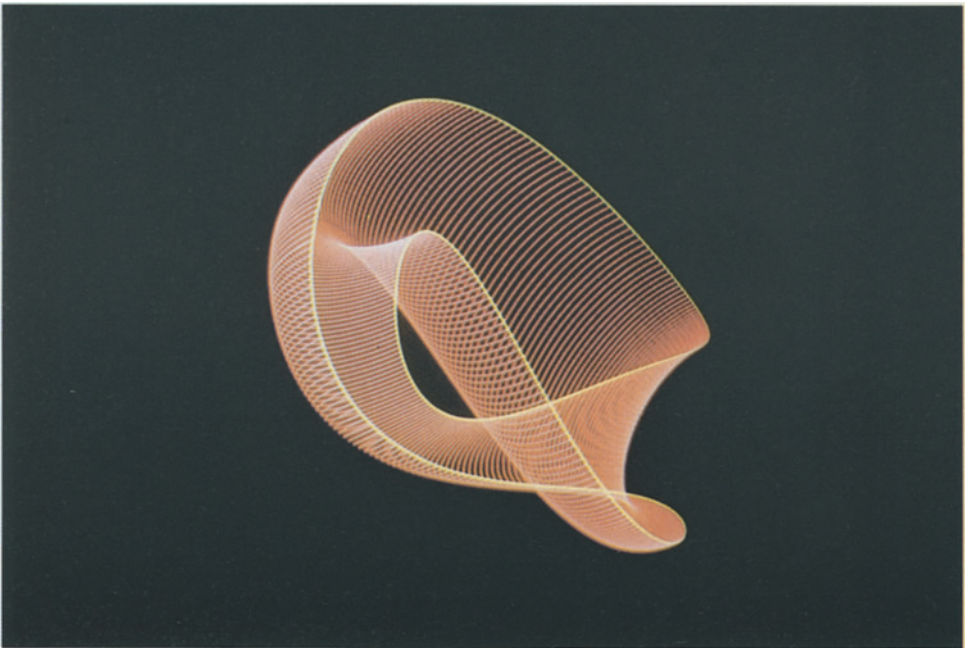


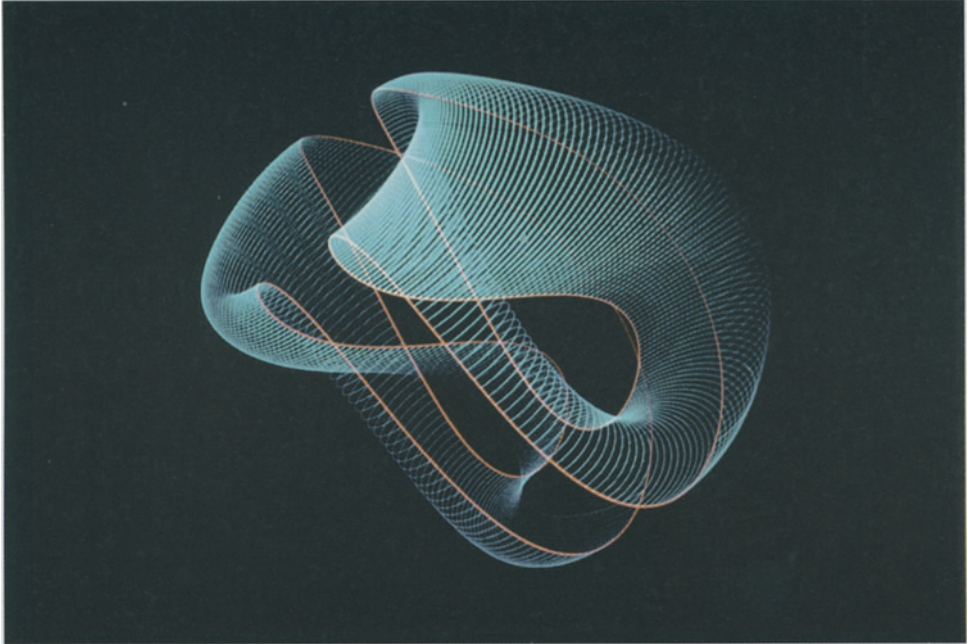




13 Möbius strip with a circle as boundary

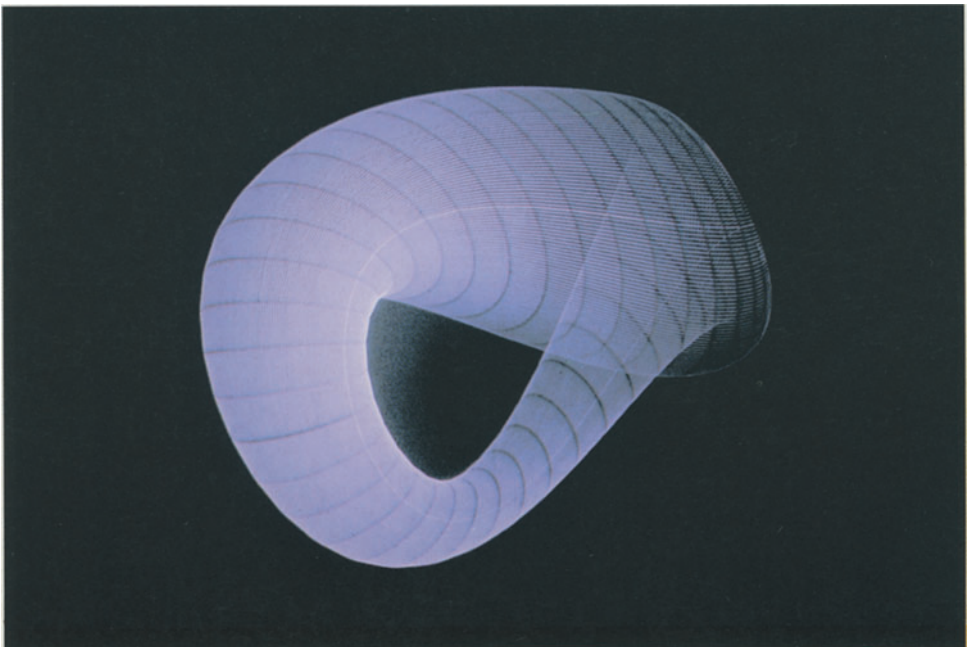
14 Immersed Möbius strip



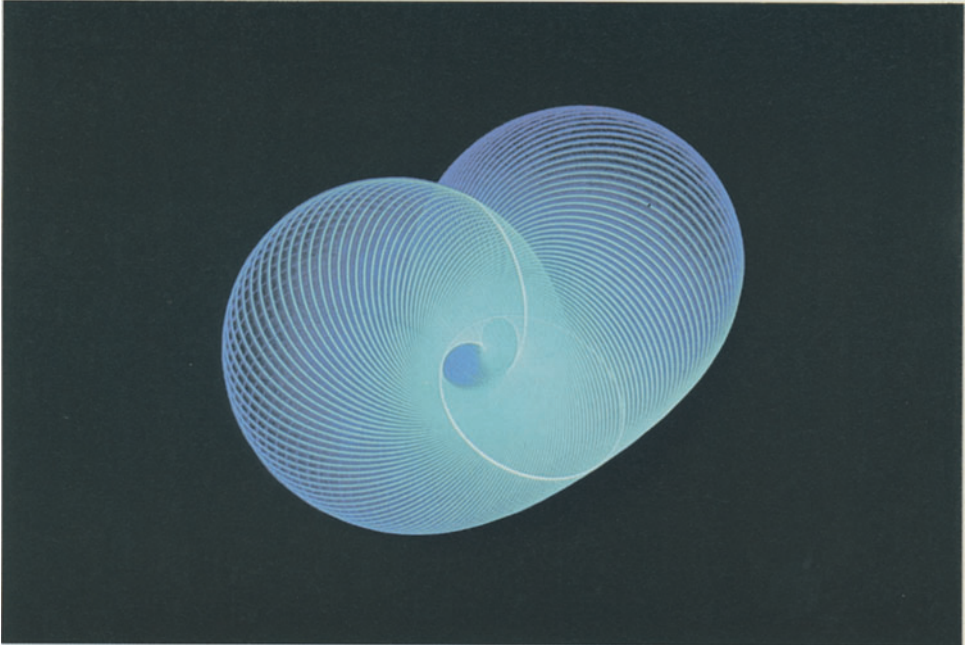


15 Gluing two Möbius strips along their boundaries

16 Klein bottle

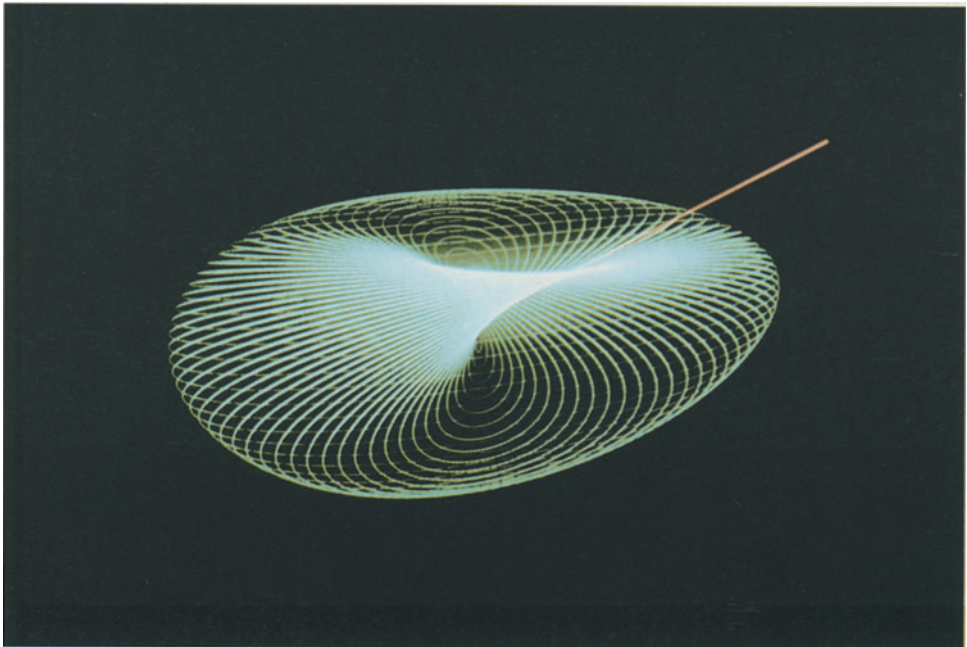


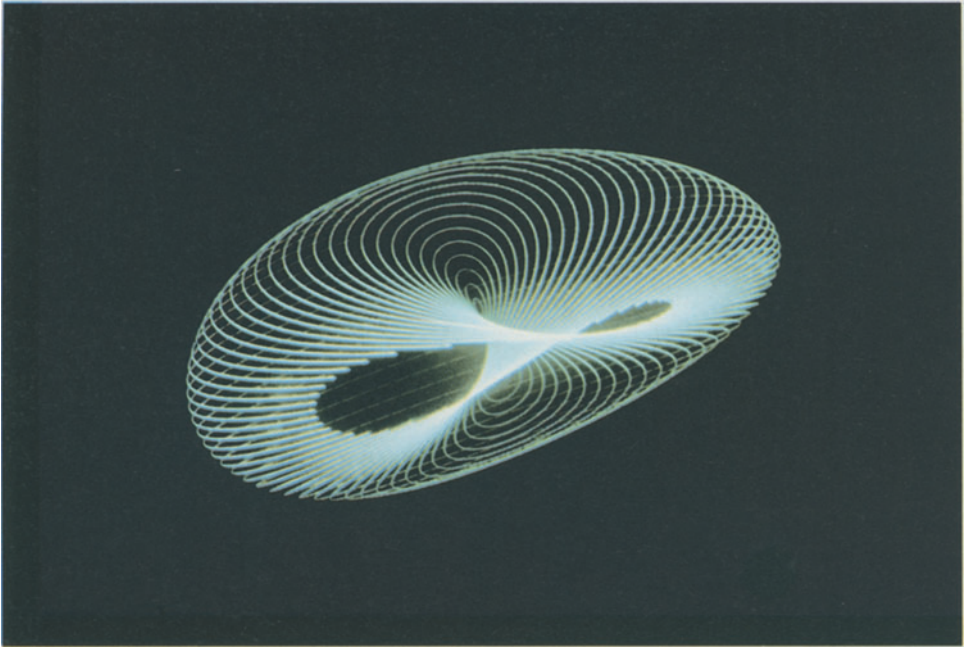




17 Klein bottle obtained by gluing two Möbius strips with circular boundary

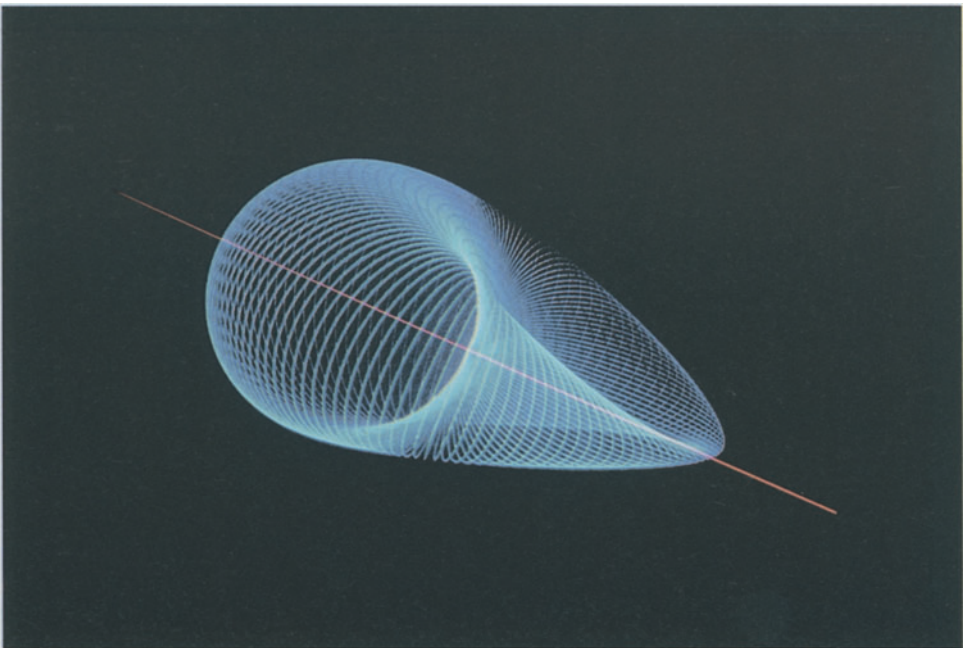
18 Steiner cross-cap

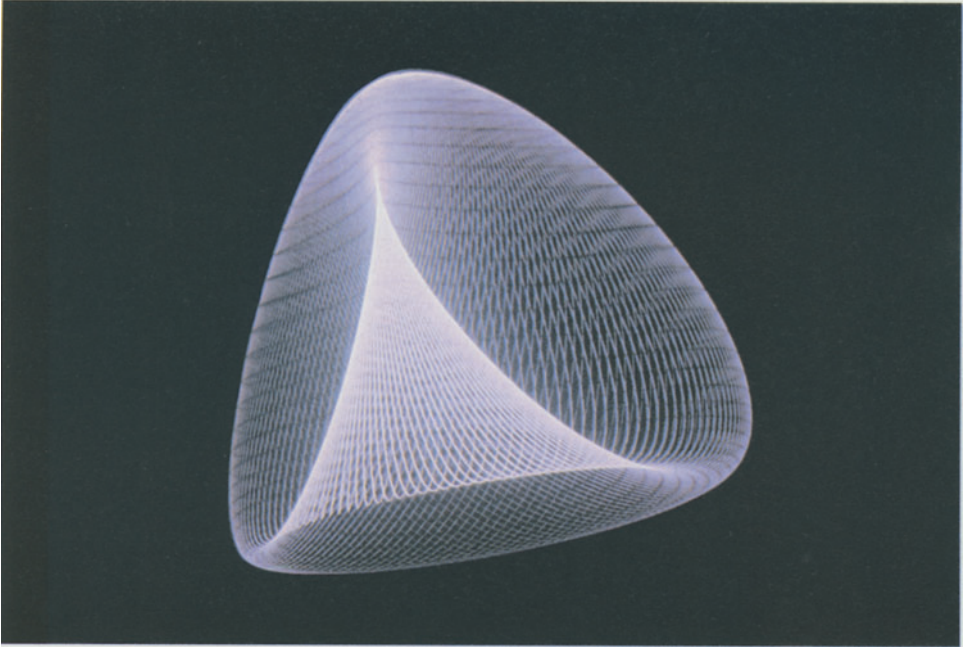




19 Steiner cross-cap with a window

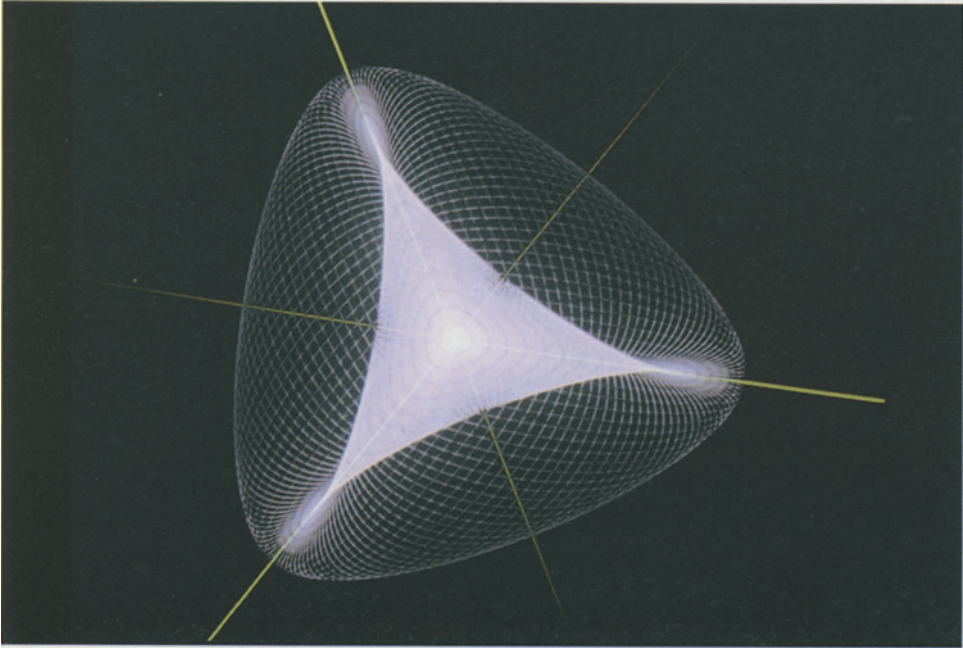
20 Steiner cross-cap



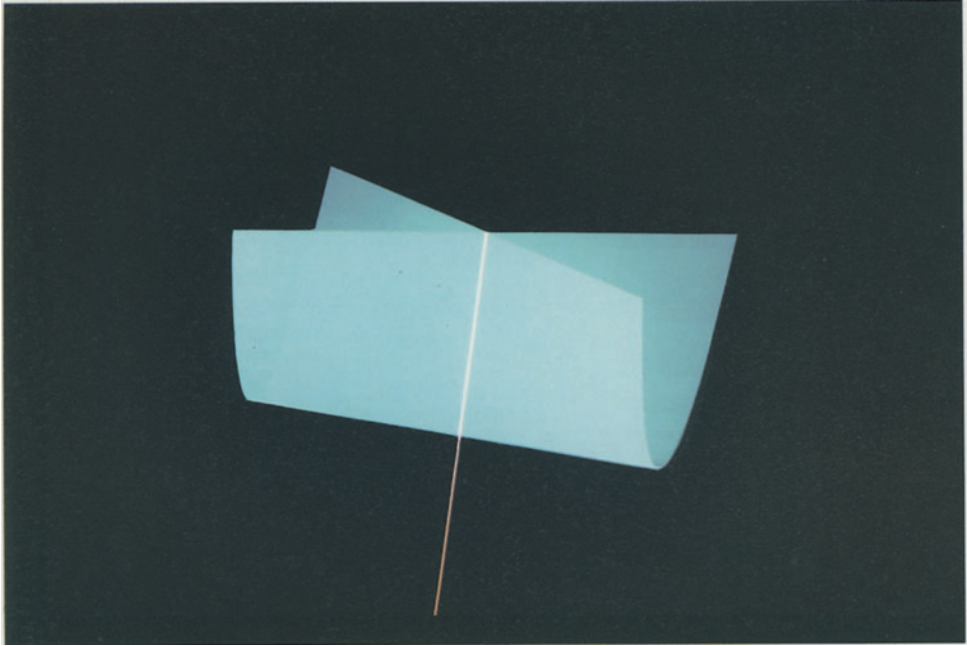


21 Image of the projective plane in the Roman surface

22 Roman surface

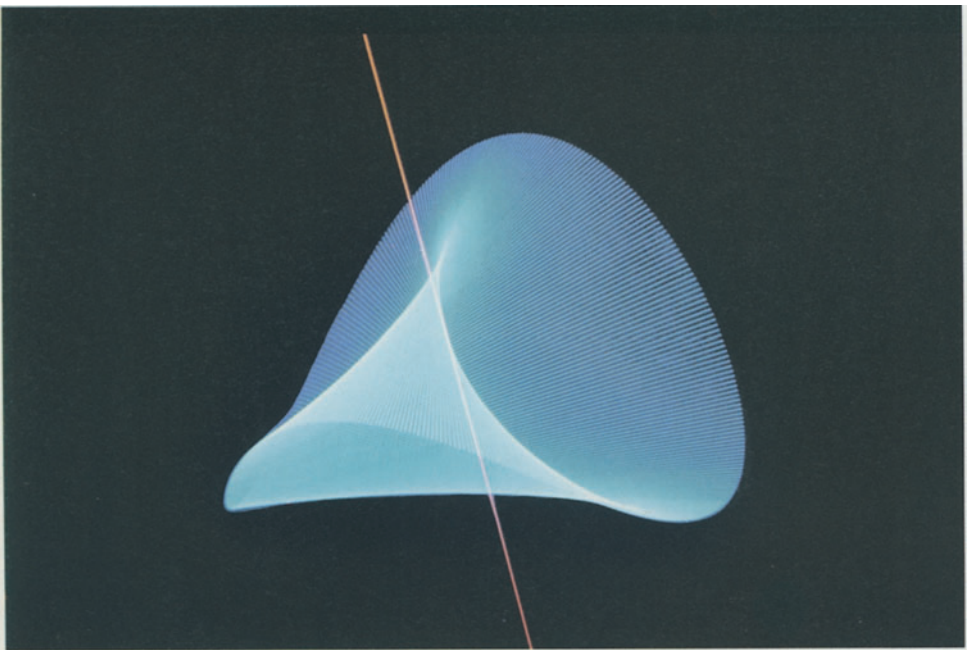


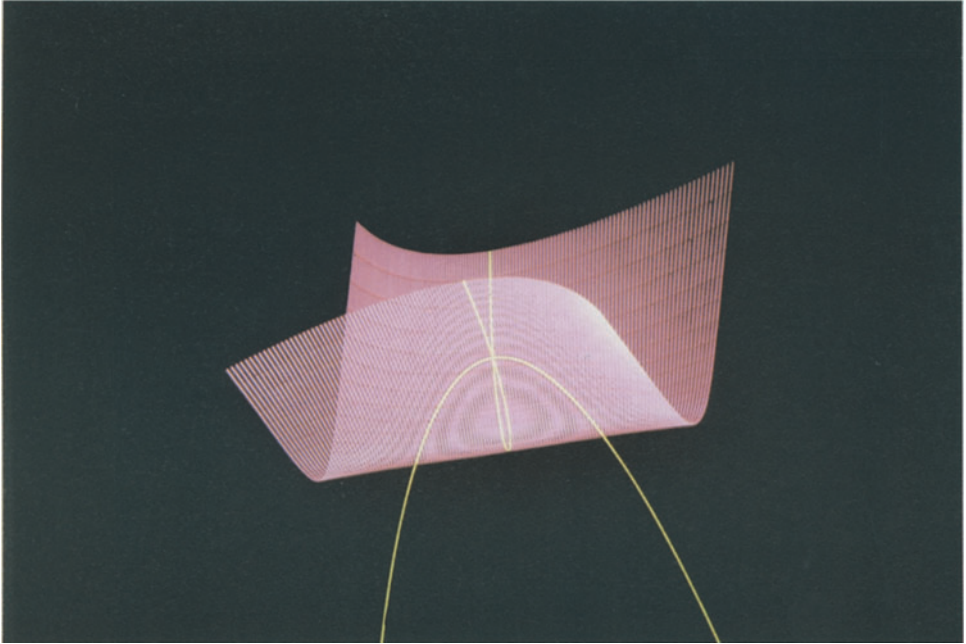




23 Whitney umbrella on the ruled cubic surface

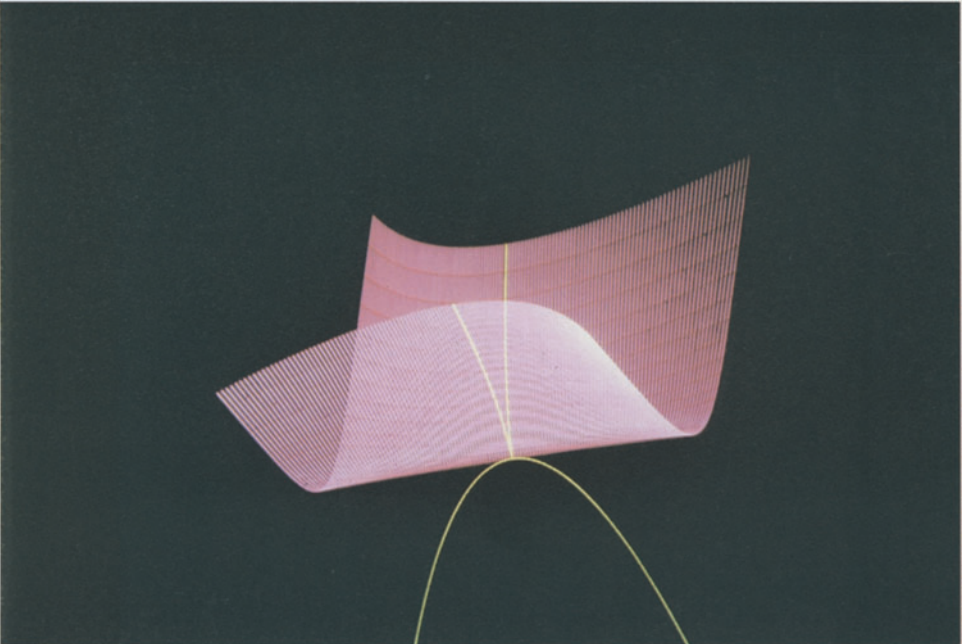
24 Plücker conoid

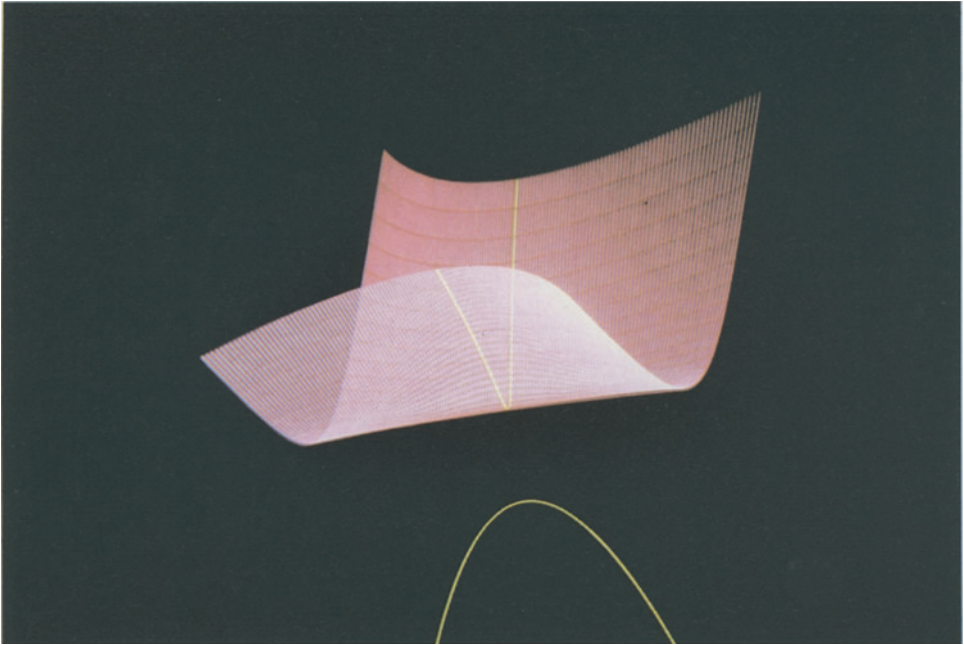




25 Elliptic confluence of two umbrellas:  $t > 0$

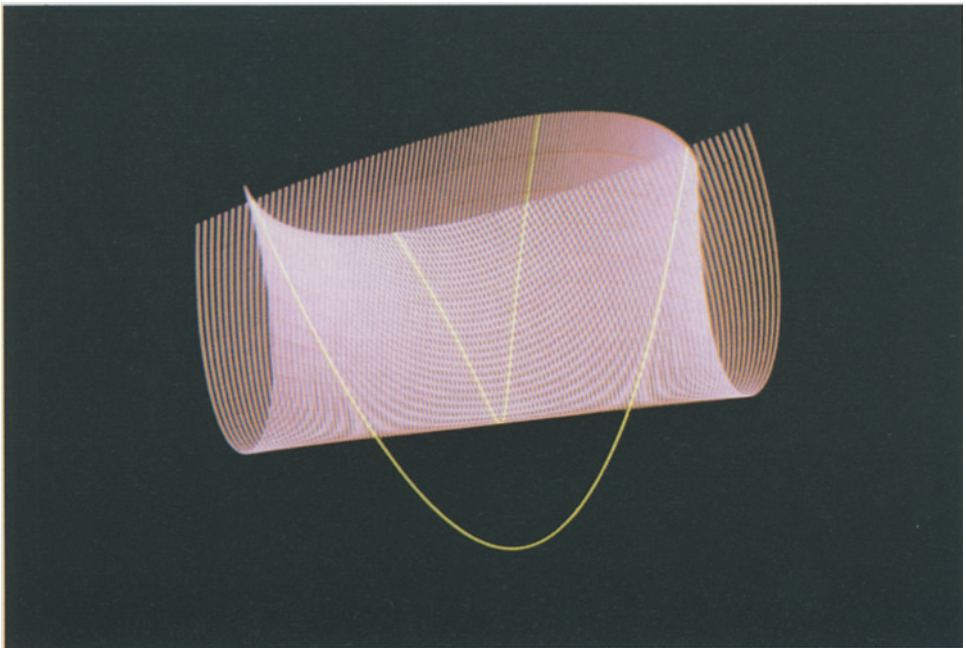
26 Elliptic confluence of two umbrellas:  $t = 0$



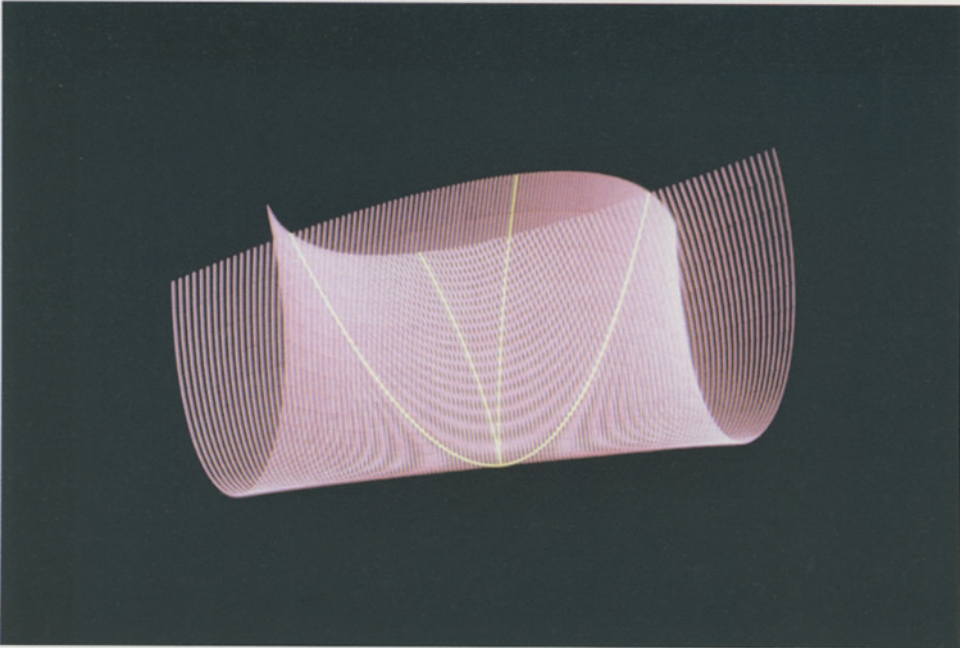


27 Elliptic confluence of two umbrellas:  $t < 0$

28 Hyperbolic confluence of two umbrellas:  $t > 0$

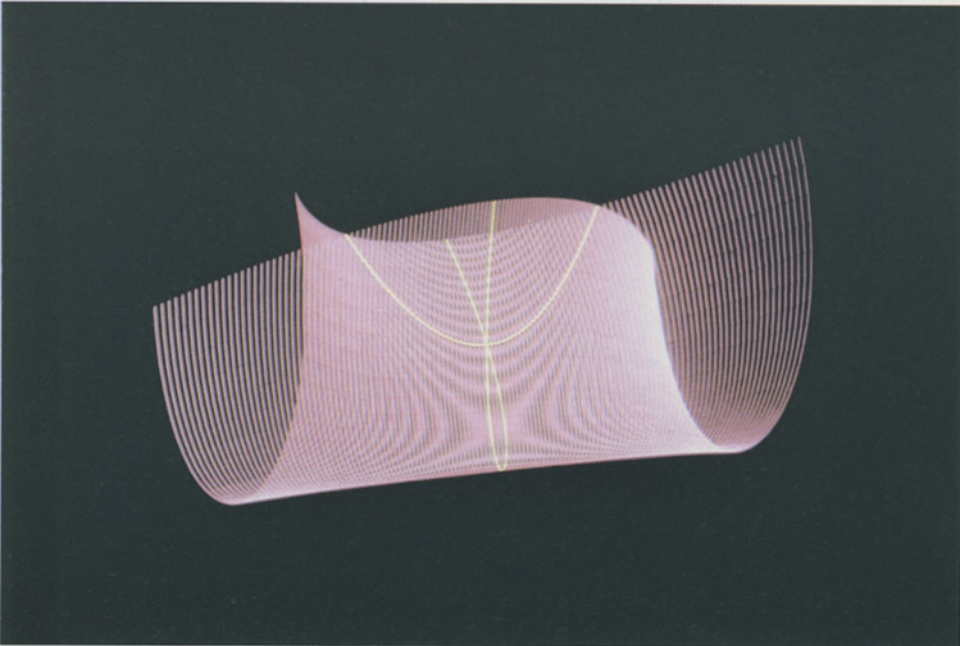


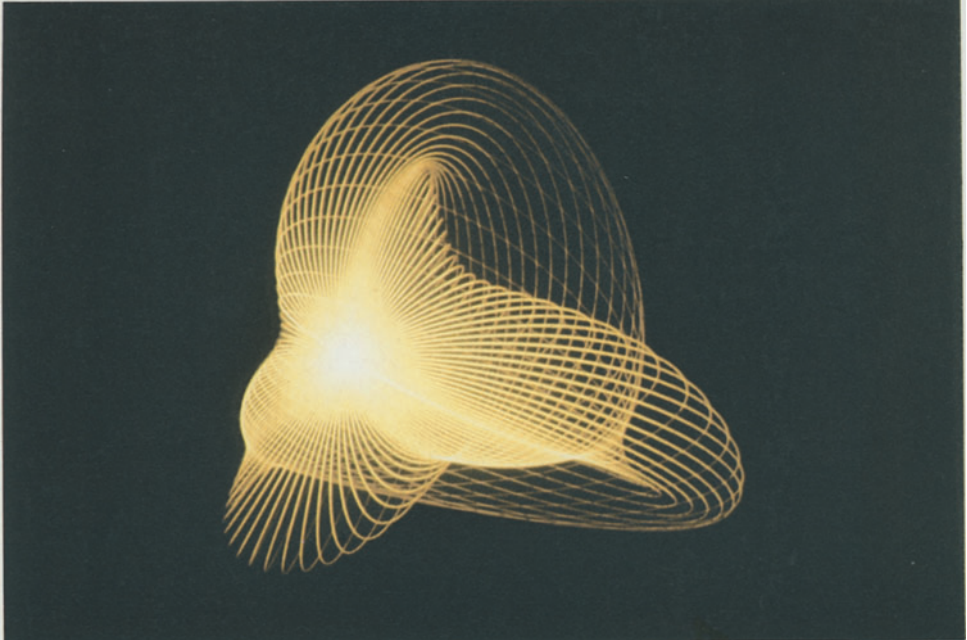




29 Hyperbolic confluence of two umbrellas:  $t = 0$

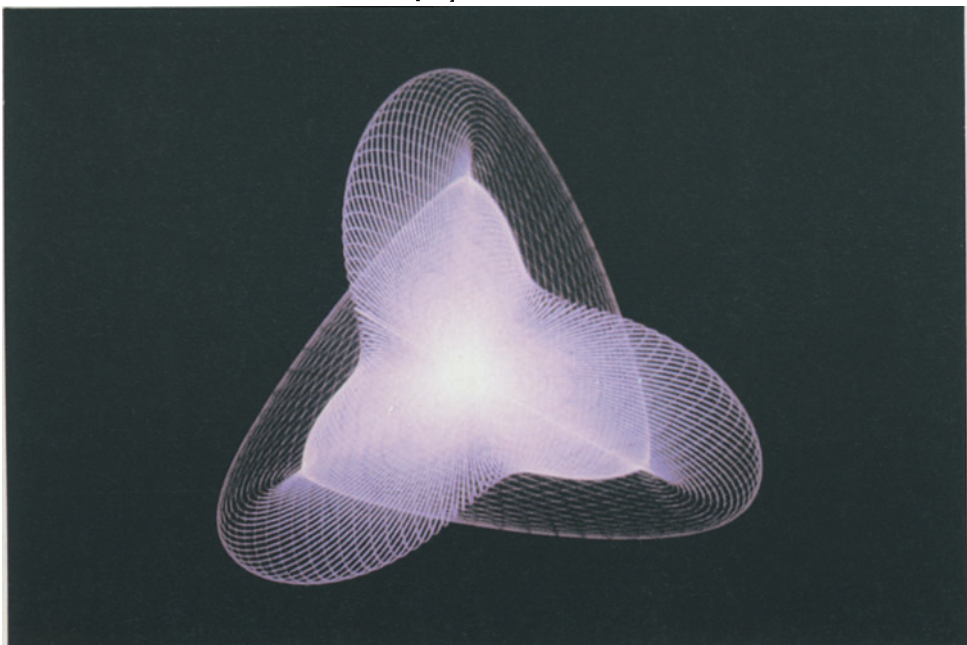
30 Hyperbolic confluence of two umbrellas:  $t < 0$



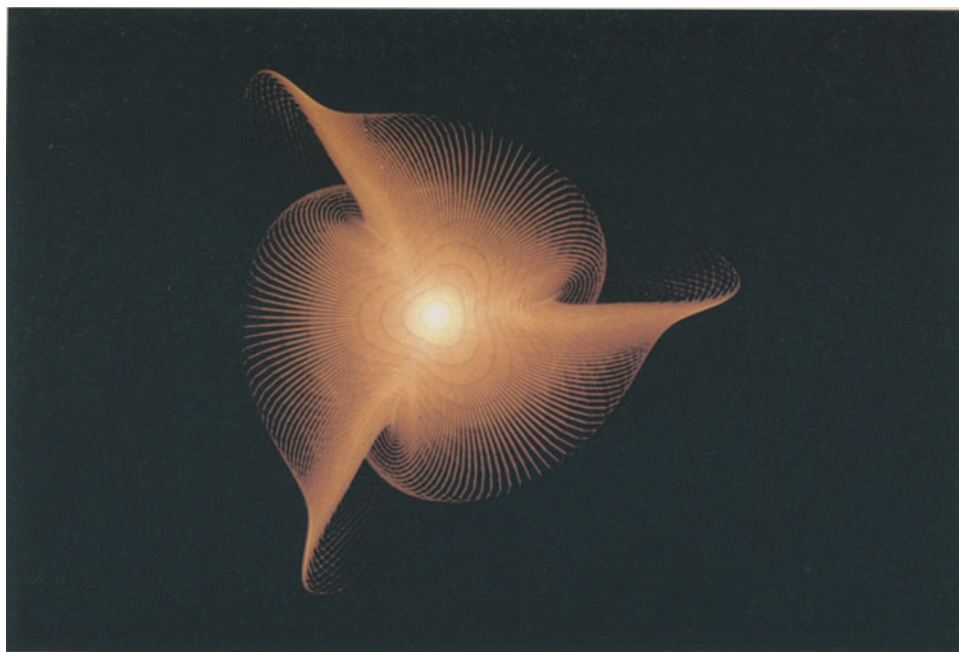


31 Boy surface according to Petit/Souriau [PE]

32 Boy surface according to Petit/Souriau [PE]

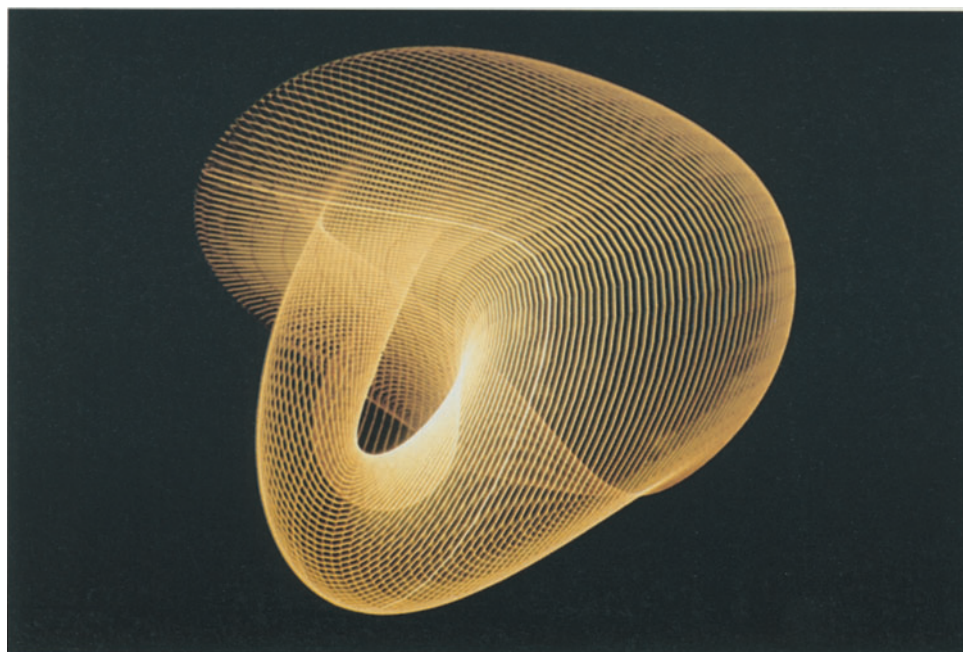


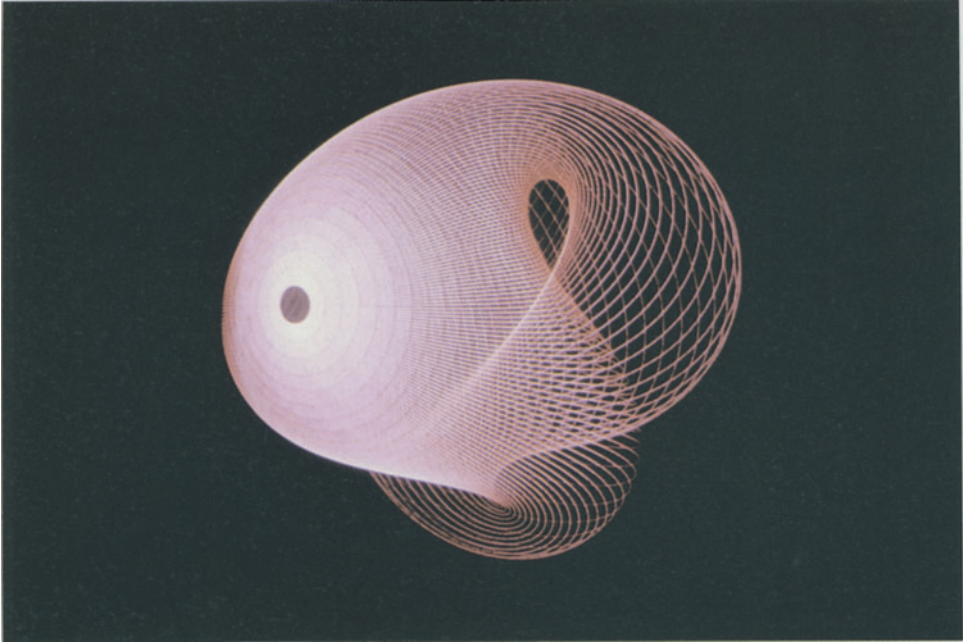




33 Boy surface according to Morin [MO2]

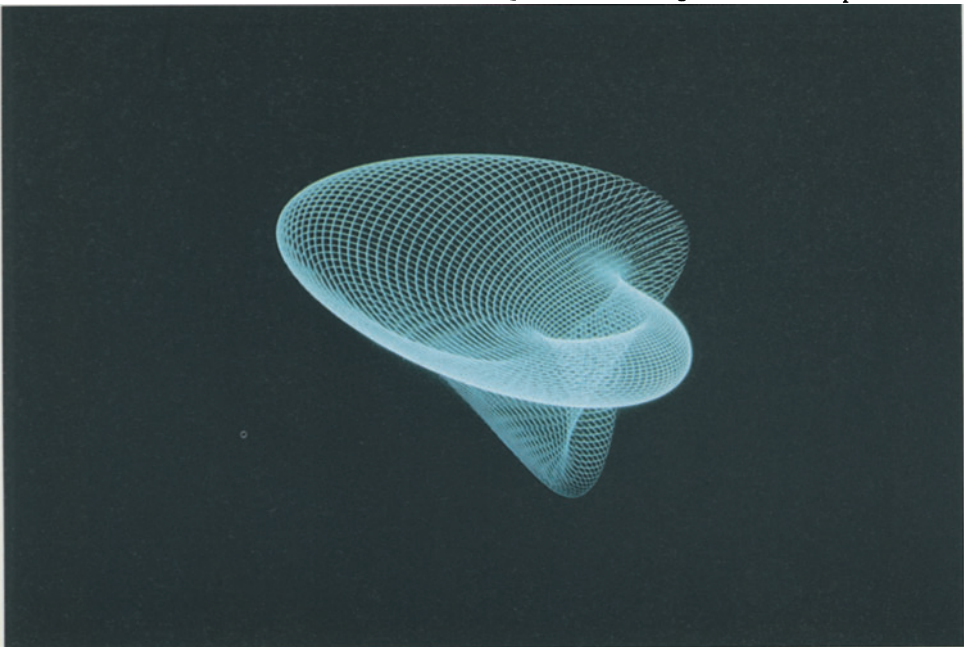
34 Boy surface according to a parametrization due to J. F. Hughes

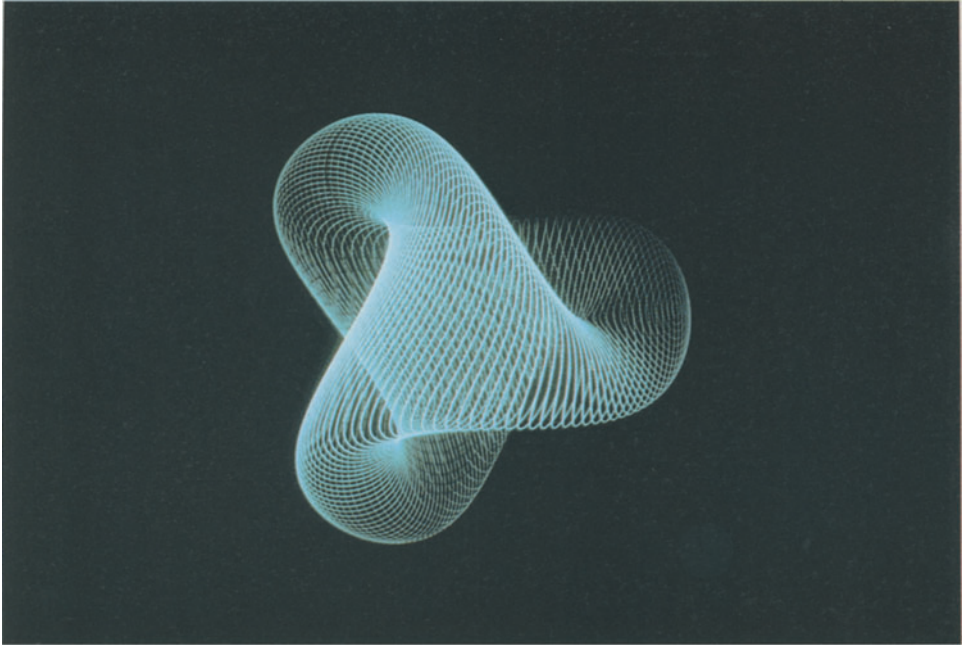




35 Boy surface according to a parametrization due to R. Bryant

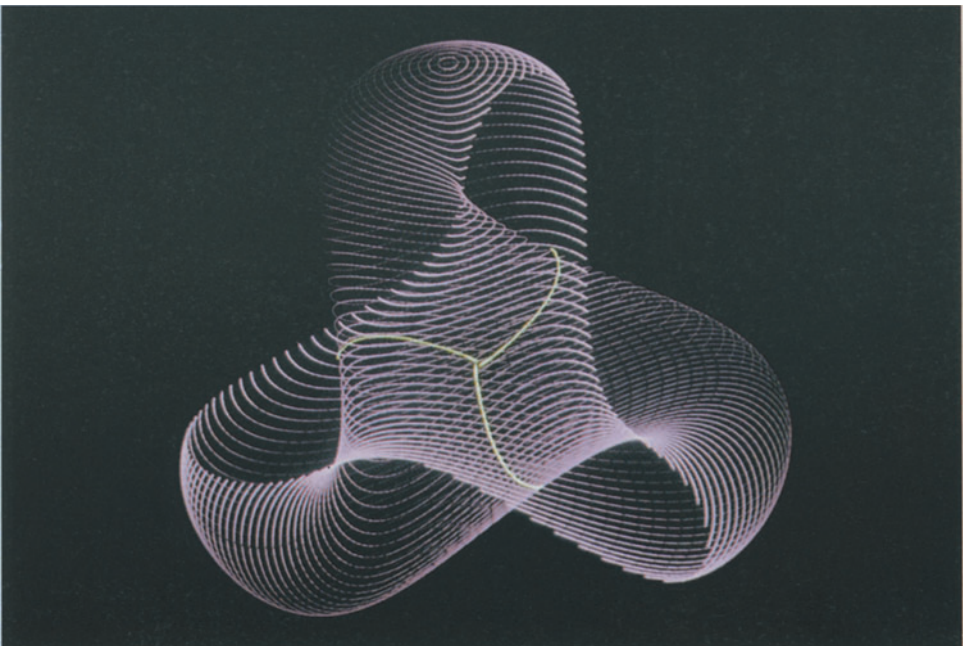
36 Boy surface parametrized by three homogeneous polynomials of degree four on the sphere



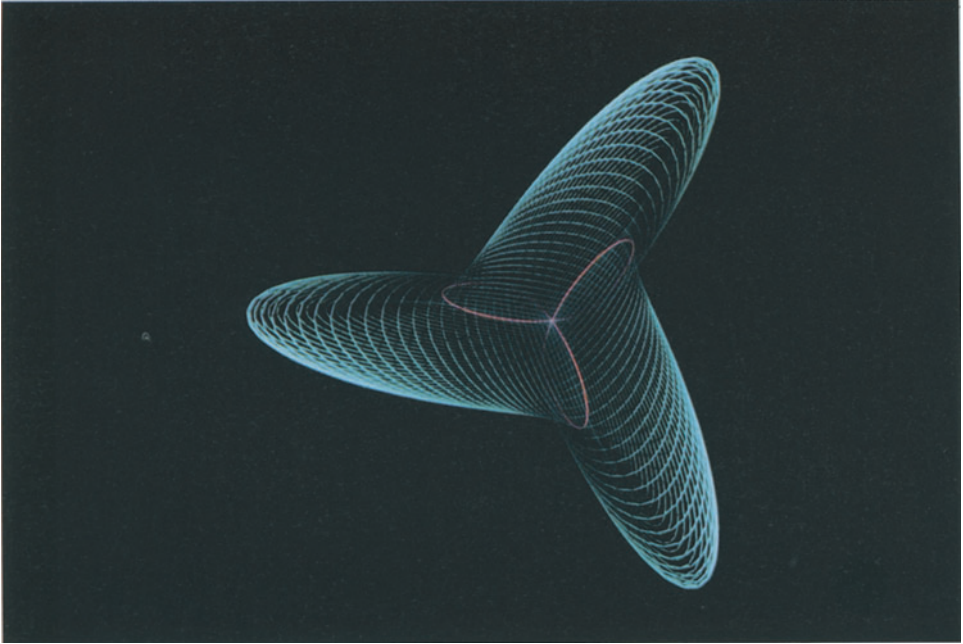


37 Boy surface parametrized by three homogeneous polynomials of degree four on the sphere

38 Boy surface parametrized by three homogeneous polynomials of degree four on the sphere with a window

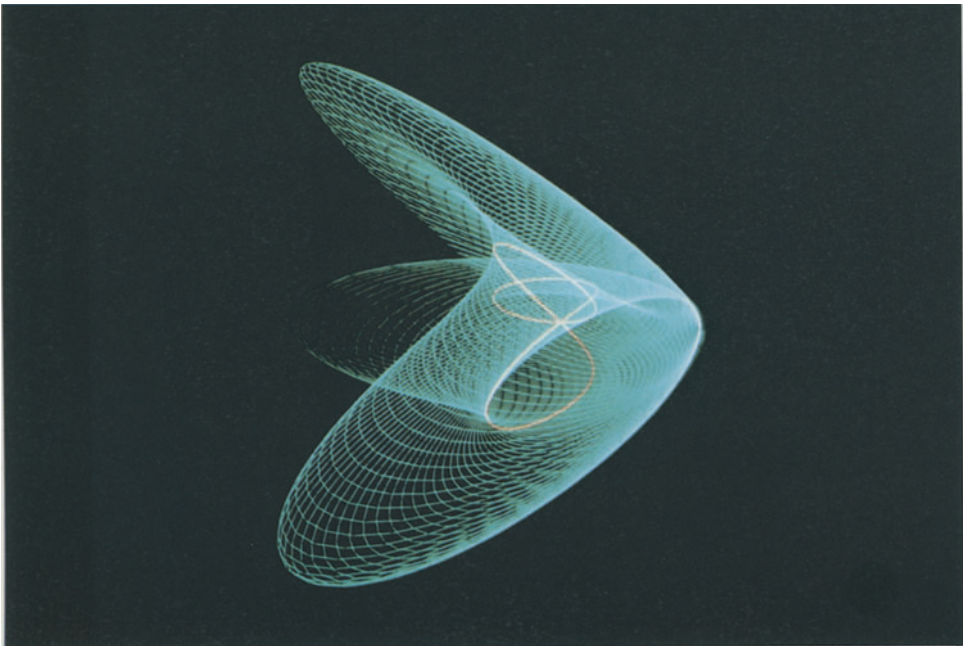


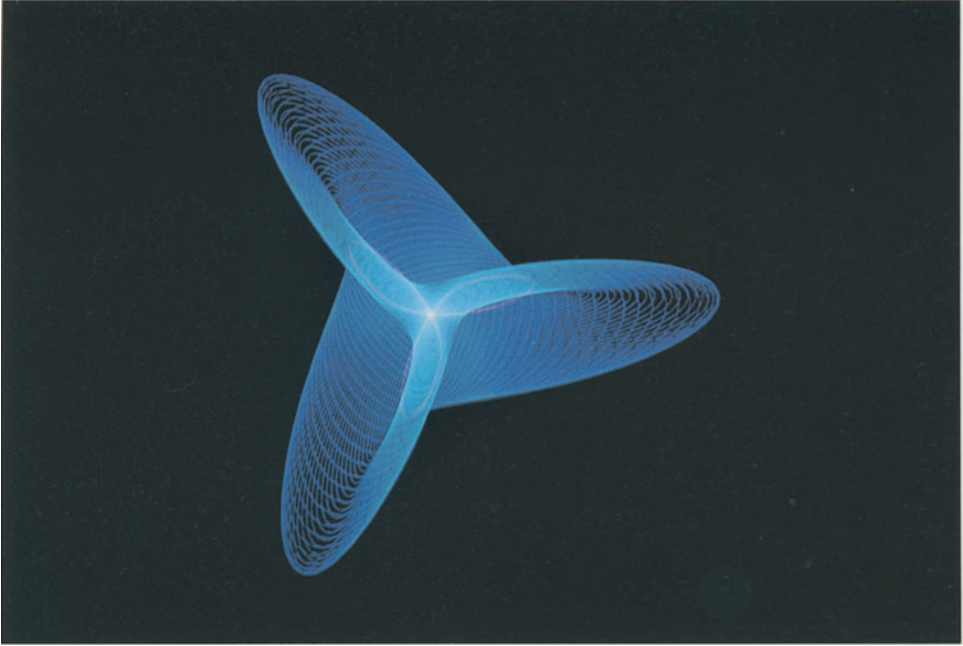




39 Boy surface of degree six

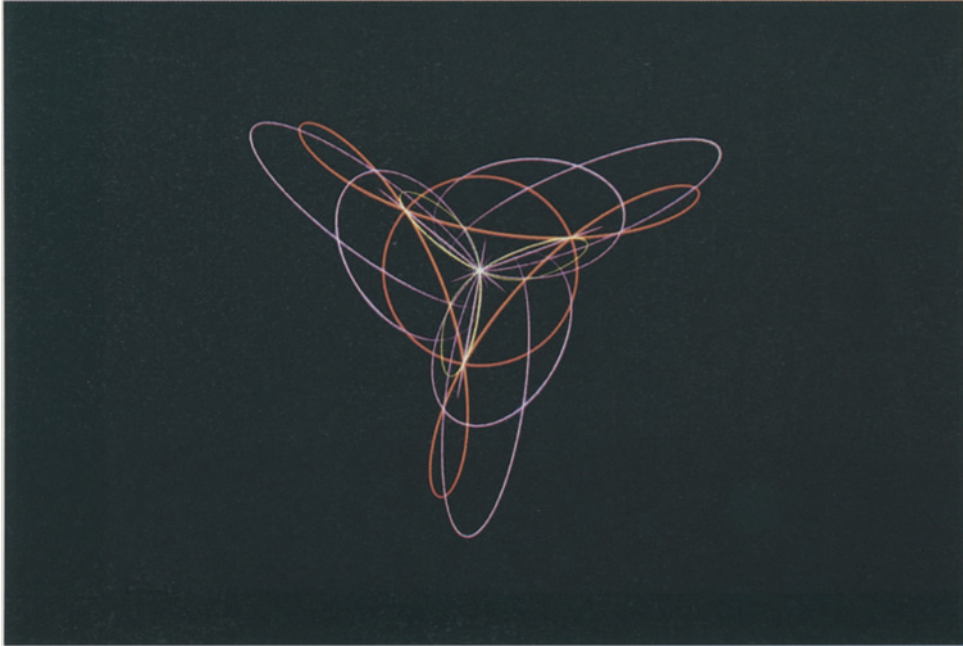
40 Boy surface of degree six

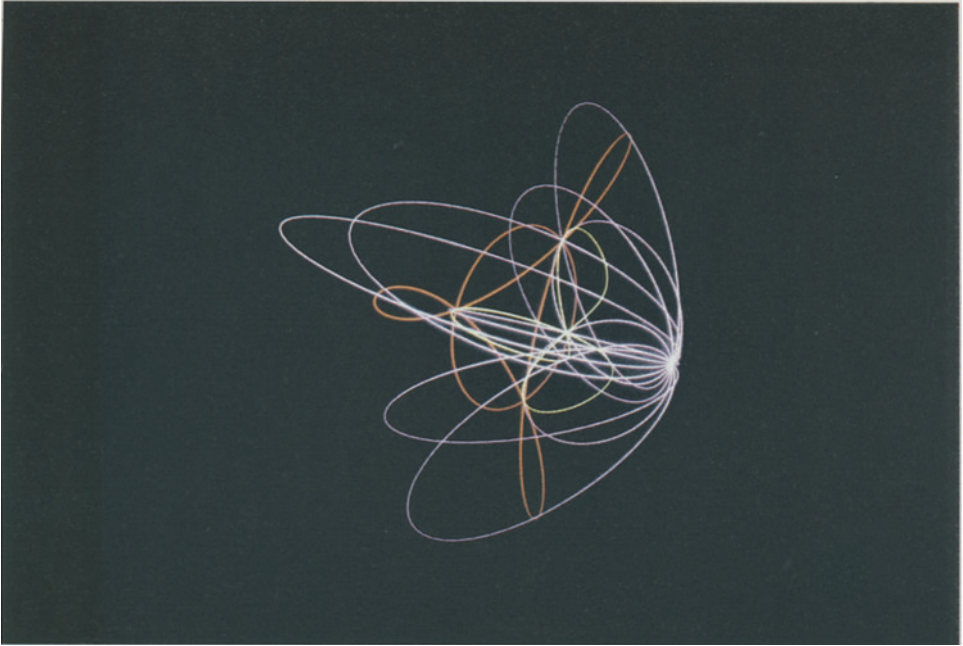




41 Boy surface of degree six

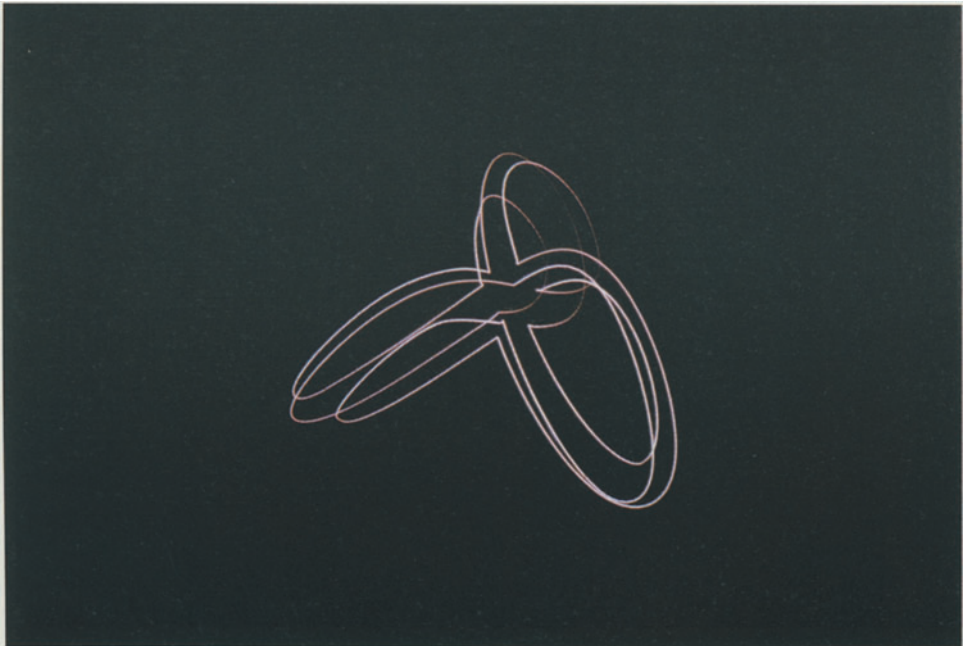
42 Curves of construction of the Boy surface of degree six



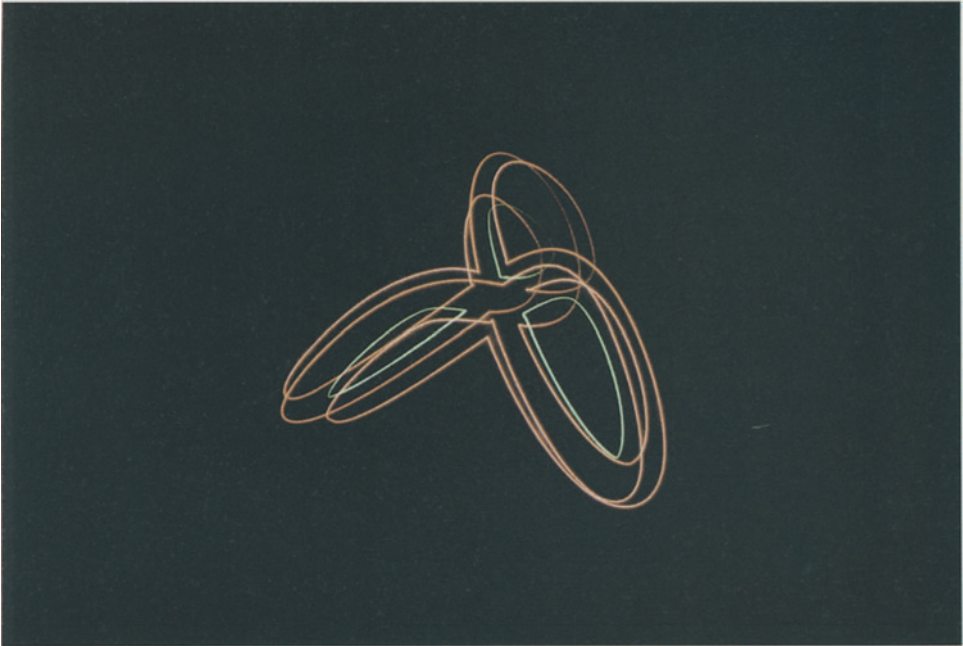


43 Curves of construction of the Boy surface of degree six

44 Boundary component of a neighborhood of the self-intersection set of the Boy surface

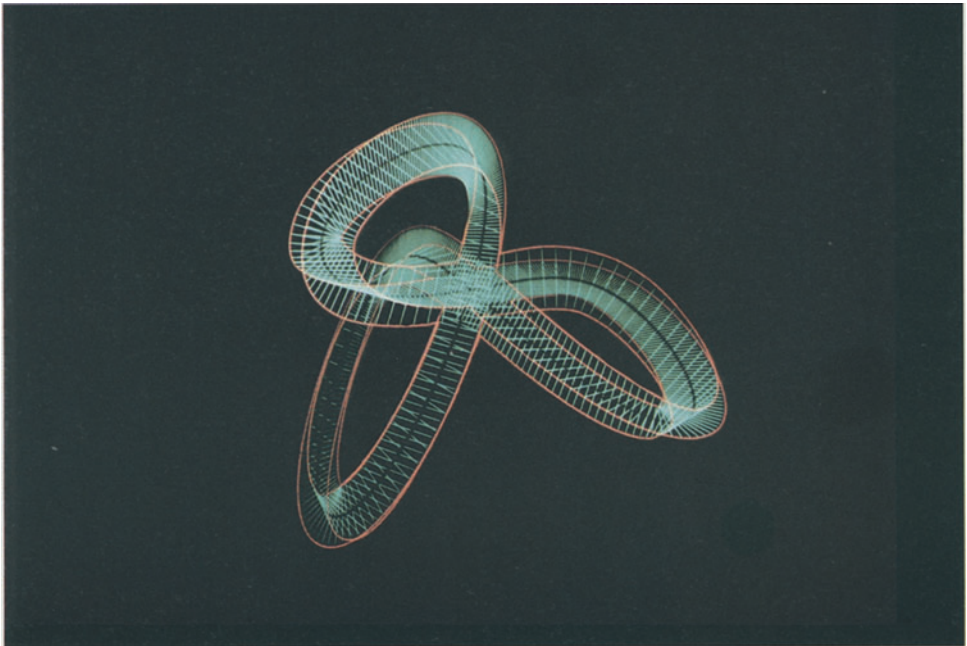


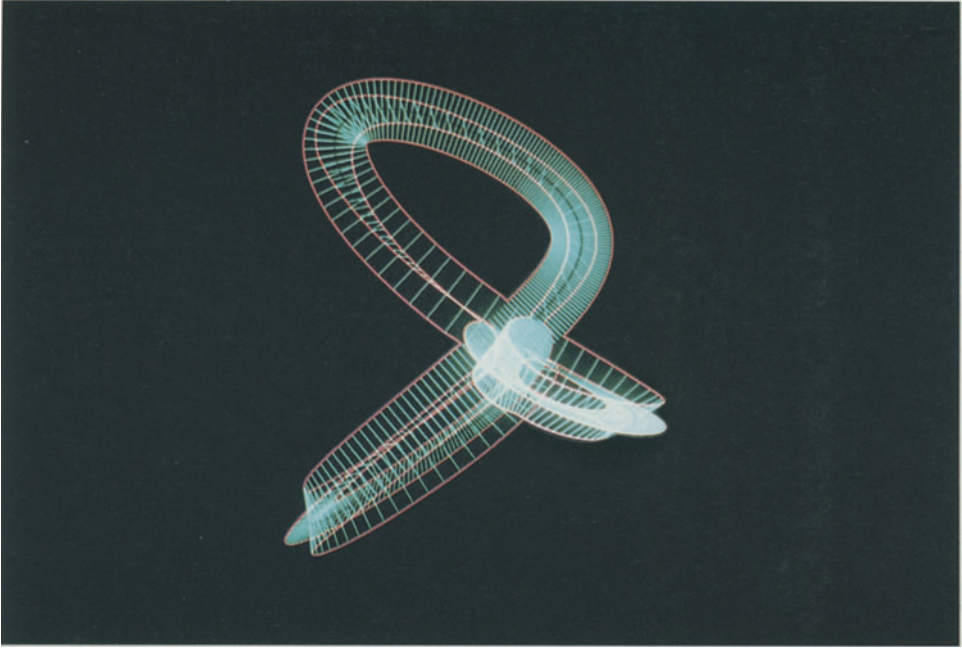




45 Boundary of a neighborhood of the self-intersection set of the Boy surface

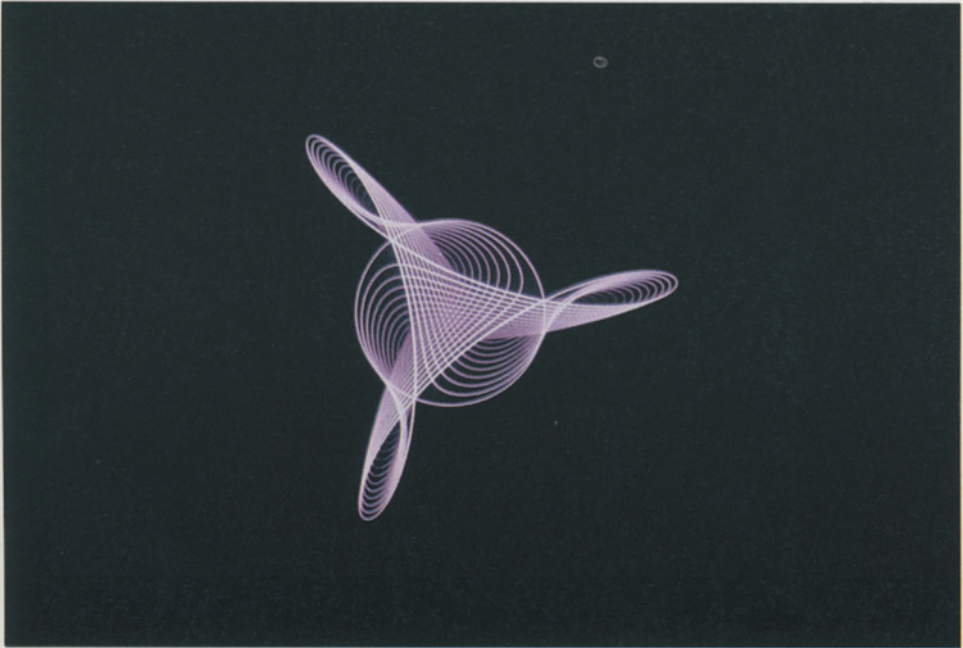
46 Neighborhood of the self-intersection set of the Boy surface



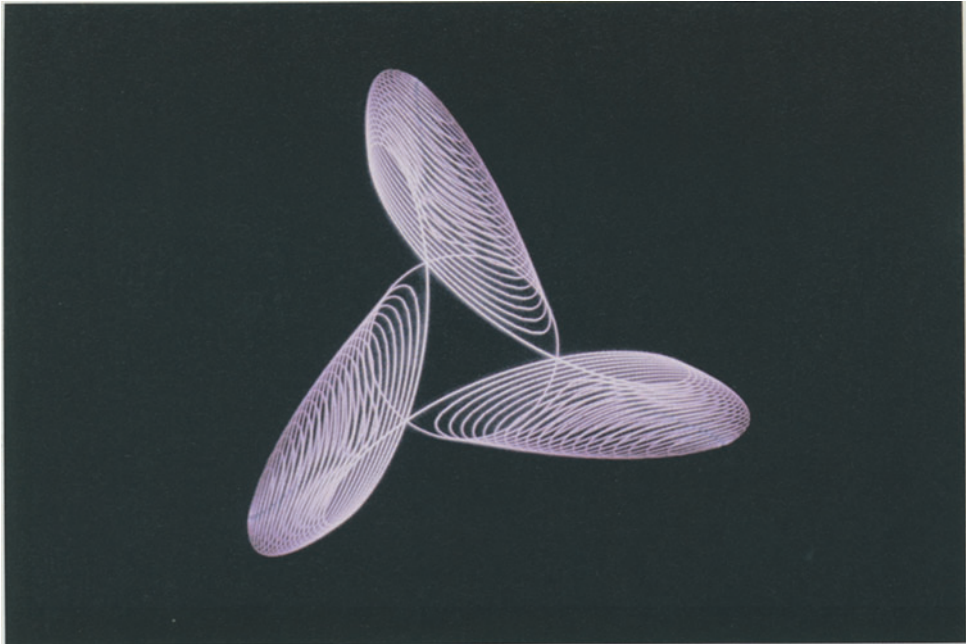


47 Neighborhood of the self-intersection set of the Boy surface

48 Level curves of the Boy surface situated between the plane of saddles and the pole

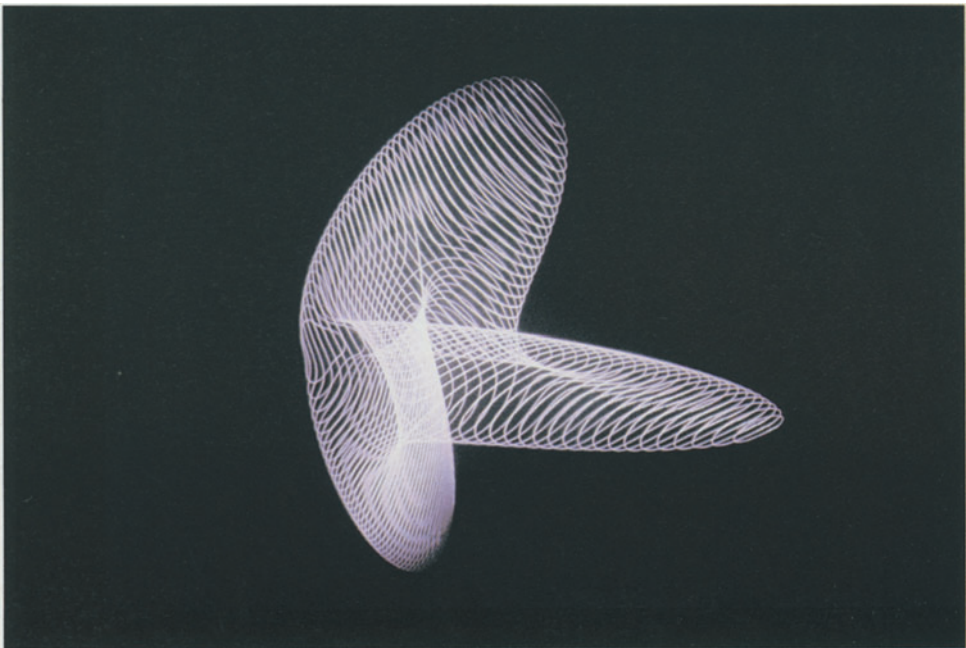


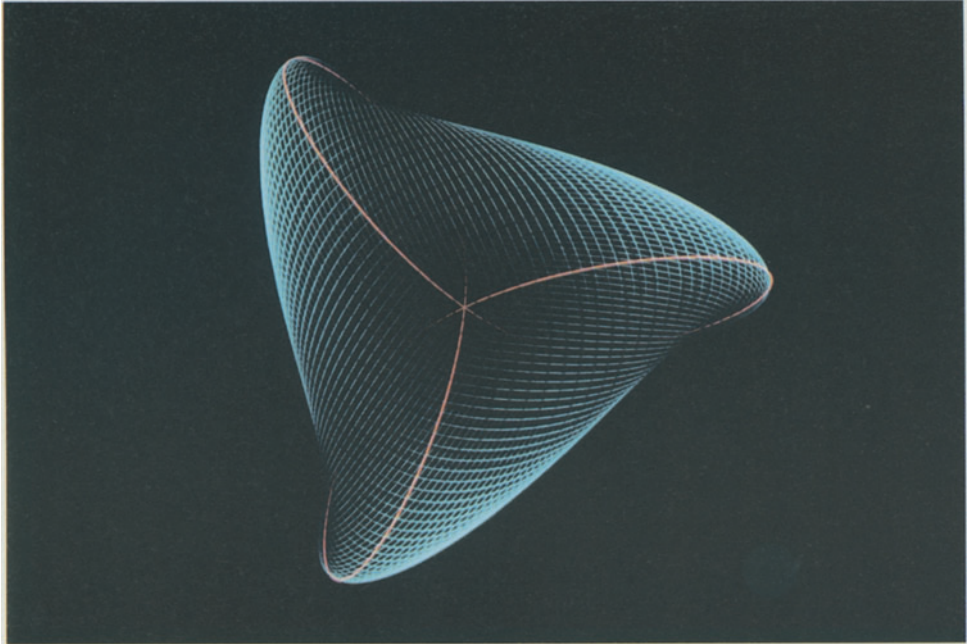




49 Level curves of the Boy surface situated between the plane of saddles and the plane of minima

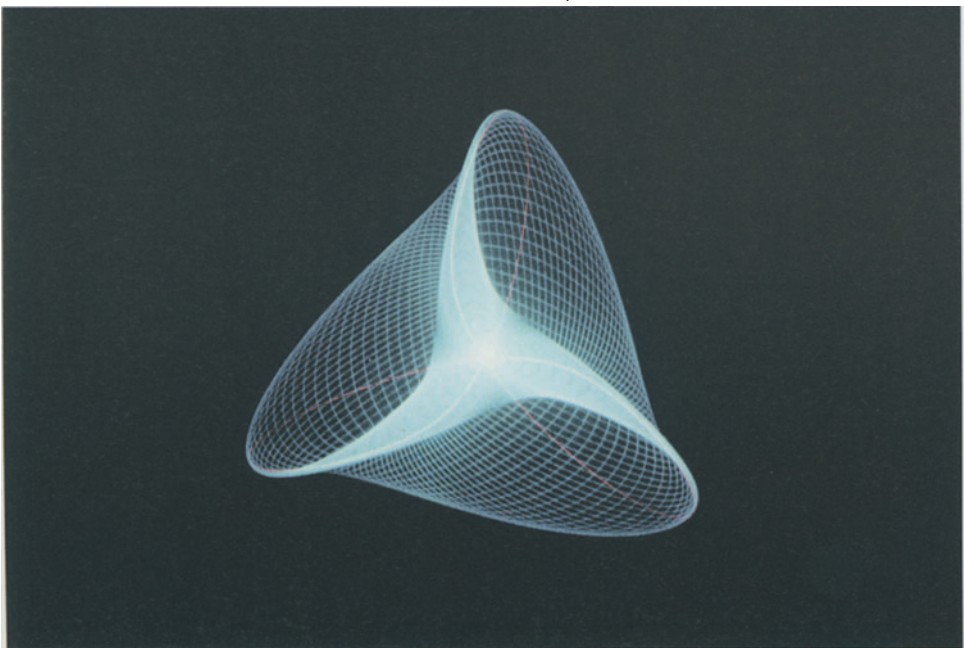
50 Level curves of the Boy surface

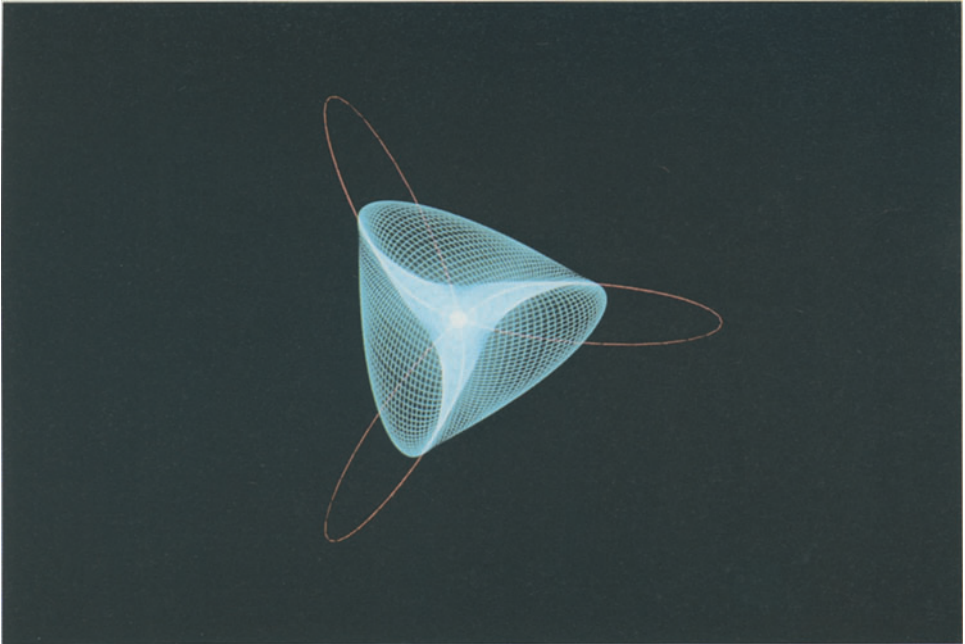




51 Confluence of umbrellas on the Roman surface  $d = 1/\sqrt{3}$

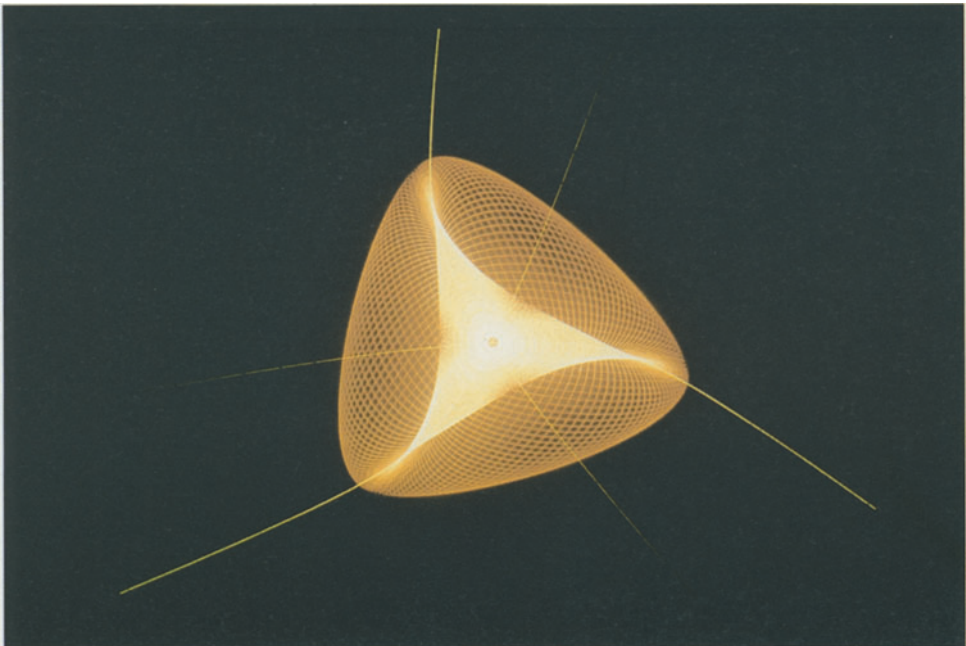
52 Confluence of umbrellas on the Roman surface  $d = 1/\sqrt{3}$



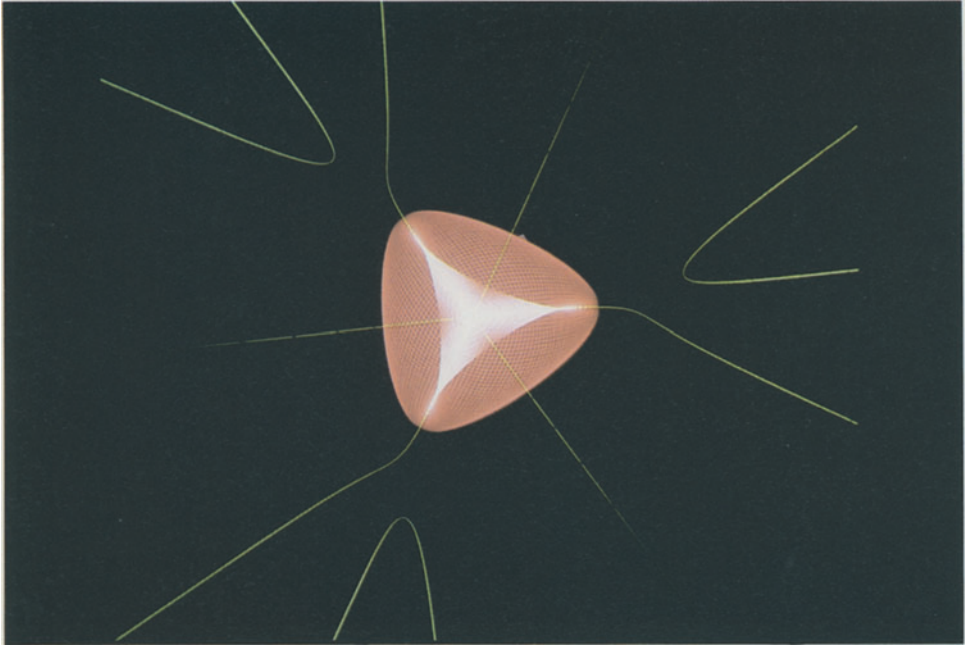


53 Deformation of the Roman surface  $d = 0.4$

54 Deformation of the Roman surface: the self-intersection curve is tangent to the plane at infinity  $d = (\sqrt{2} - 1)^2$

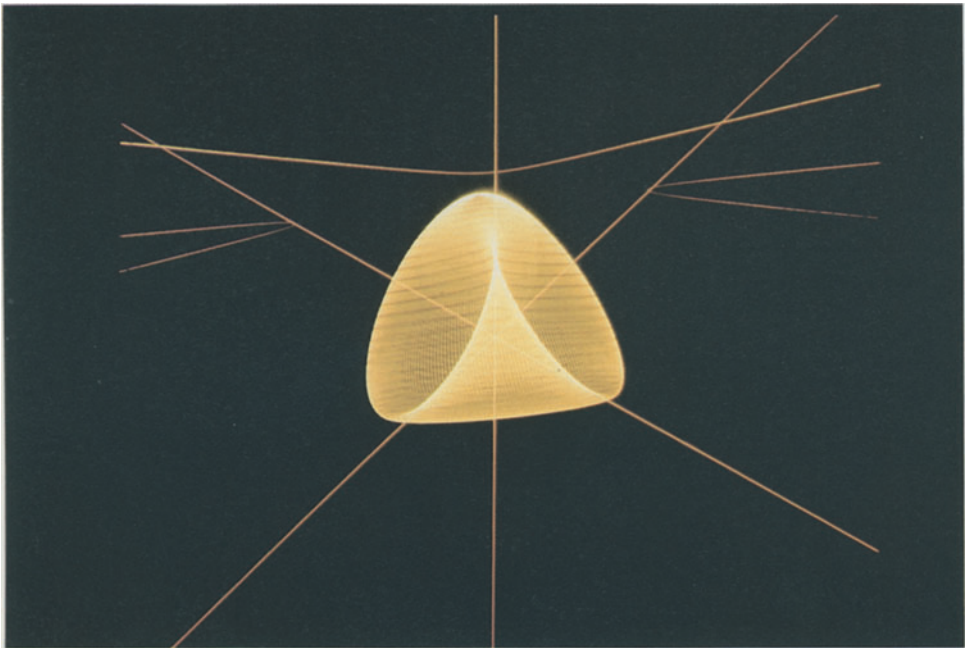


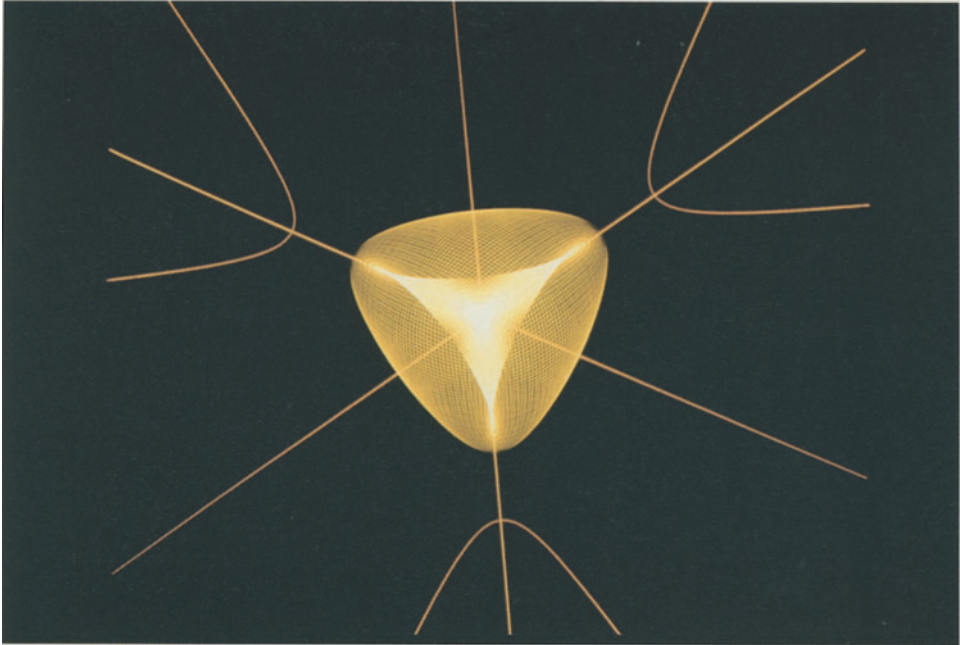




55 Deformation of the Roman surface  $d = 0.001$

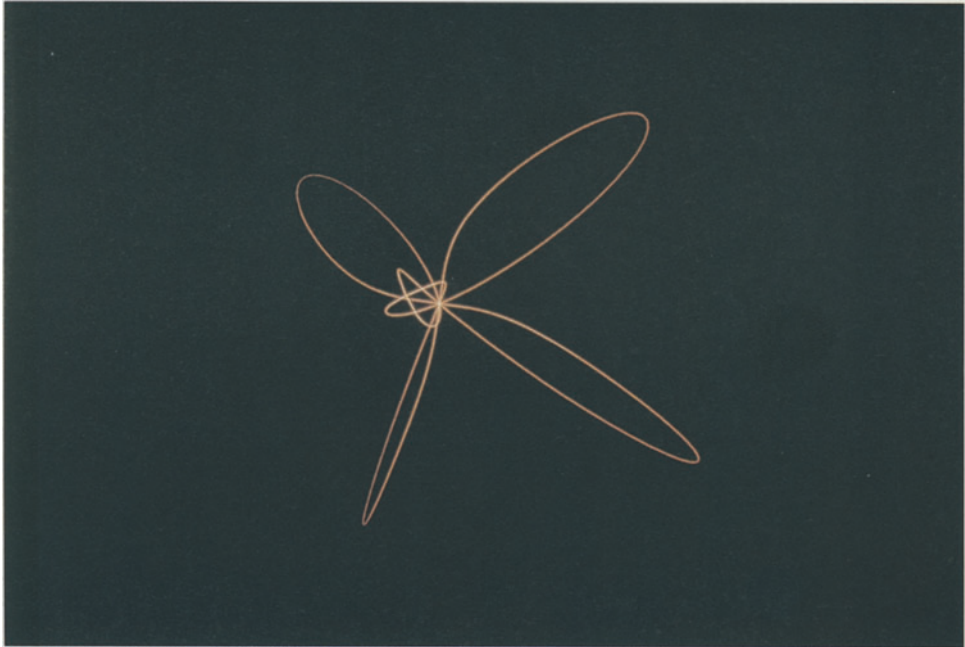
56 Beginning of the deformation of the Roman surface  $d = 0$

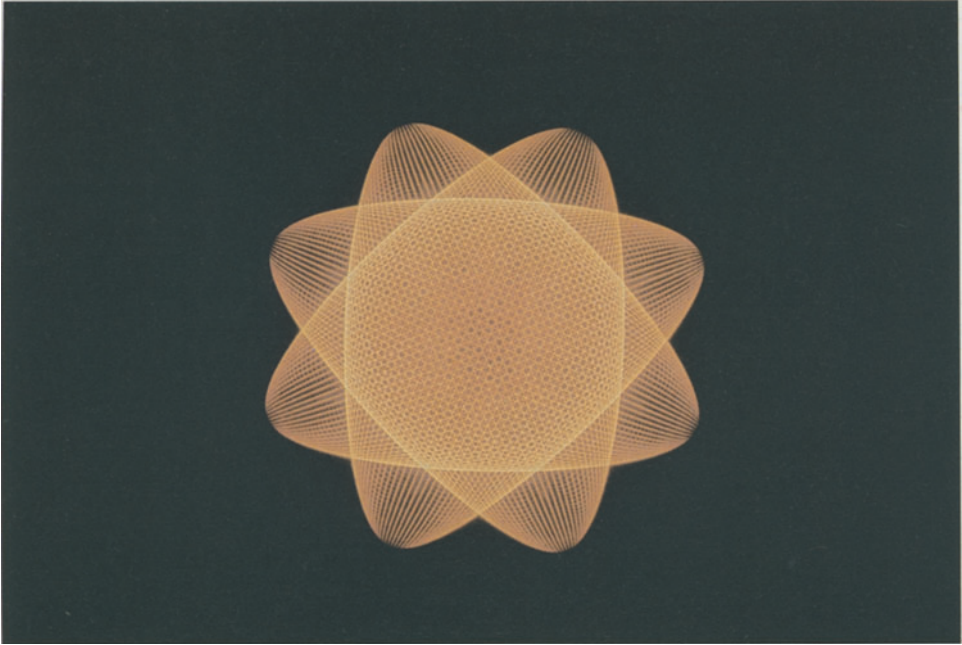




57 Beginning of the deformation of the Roman surface  $d = 0$

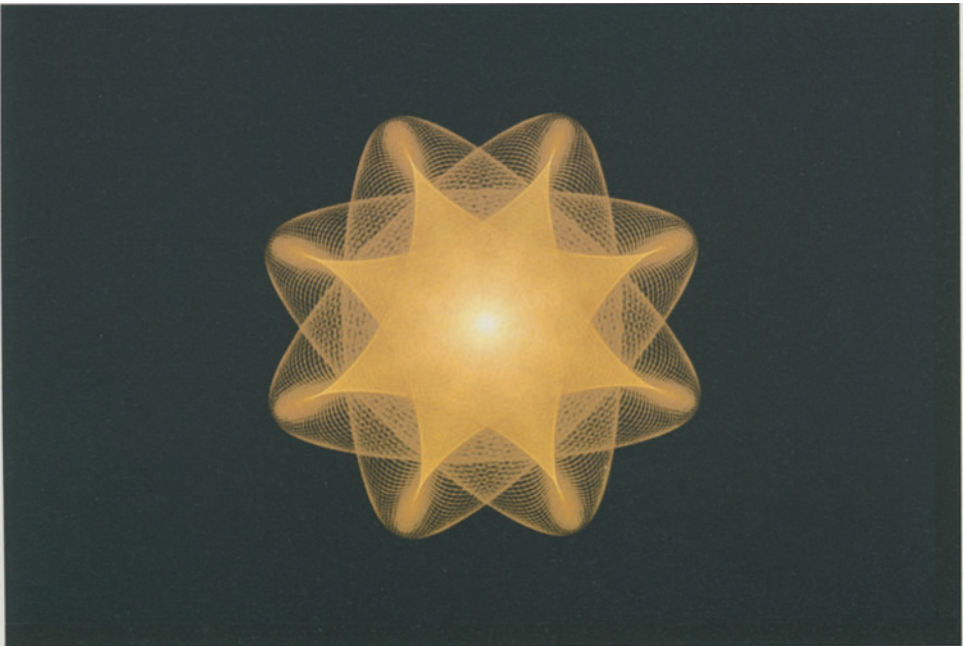
58 Self-intersection set of the halfway model



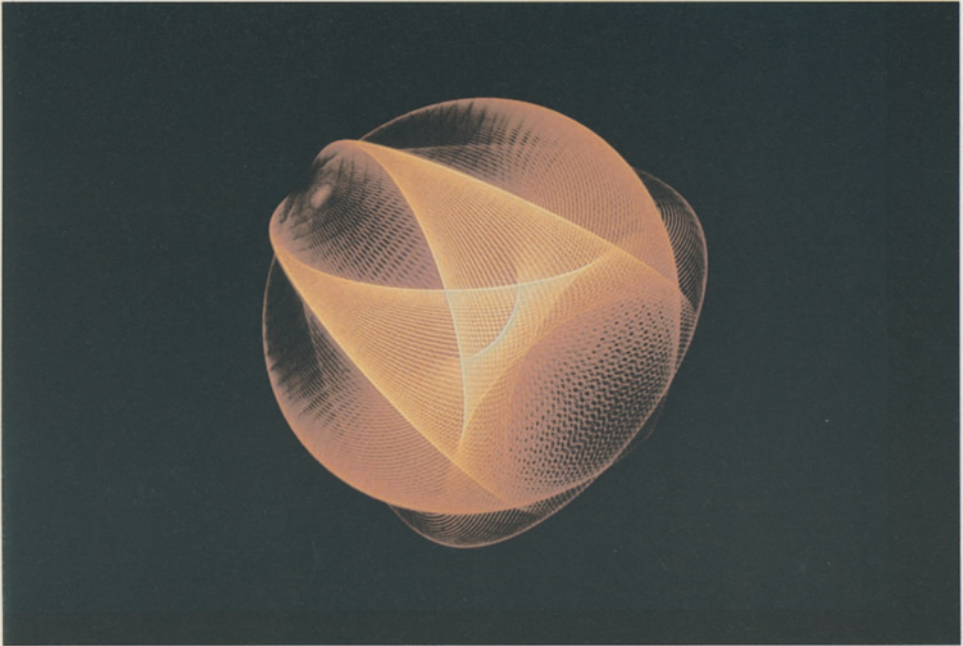


59 Surface of Roman type having an eightfold symmetry

60 Surface of Roman type having an eightfold symmetry

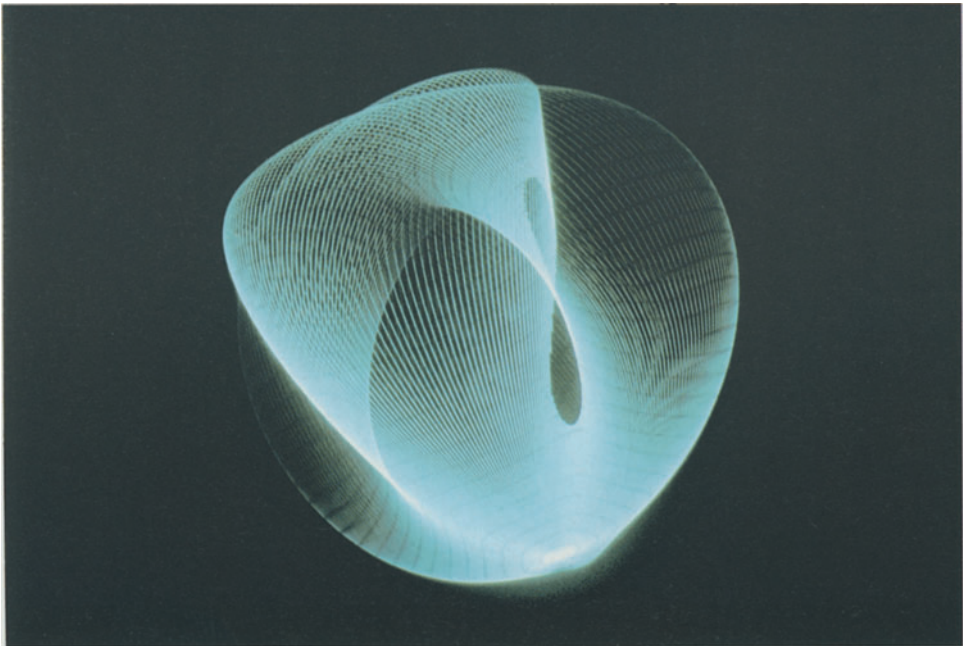


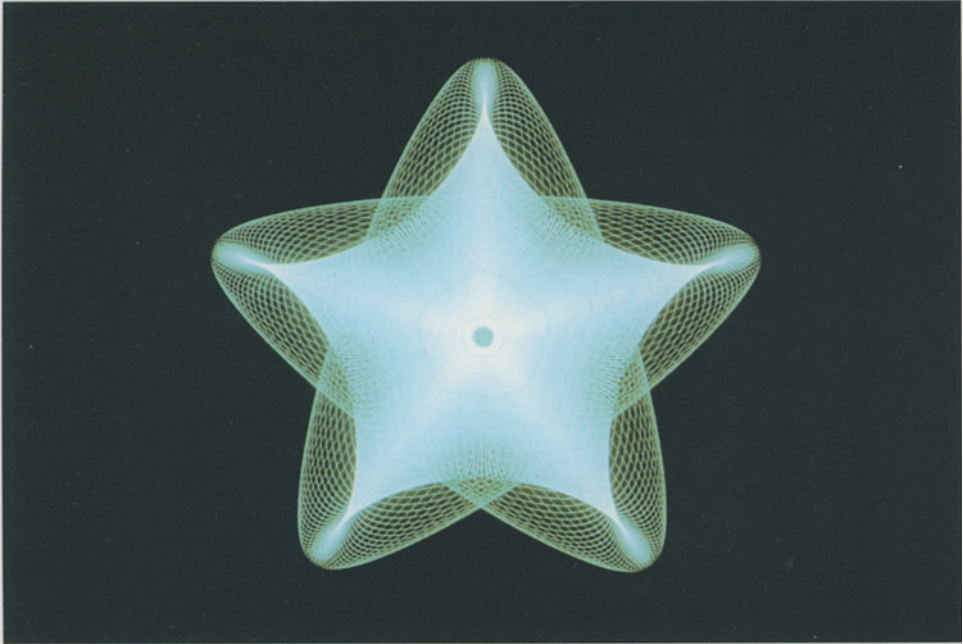




61 Surface of Roman type having an eightfold symmetry

62 Surface of Roman type having a fivefold symmetry with window





63 Surface of Roman type having a fivefold symmetry

64 Immersed projective plane having a fivefold symmetry

