Appendix

Listing of the FORTRAN-program used to draw the Boy surface and its deformations

by *Raymond Ripp*

The program was run on a PS 300 Evans-Sutherland connected to a VAX computer.
It generates the Boy surface, the generalizations with an axis of n-fold symmetry, and all the steps of the deformation giving the Roman surface.

Parameters:

G is the parameter of *deformation*
n is the parameter of *symmetry*

$g = 0$	$n = 3$	**Plates 21, 22, 56, 57**
$g = 1$	$n = 3$	**Plates 39, 40, 41, 42, 43**
$g = 1/\sqrt{3}$	$n = 3$	**Plates 51, 52**
$g = 0,4$	$n = 3$	**Plate 53**
$g = (\sqrt{2}-1)^2$	$n = 3$	**Plate 54**
$g = 1/1000$	$n = 3$	**Plate 55**
$g = 1$	$n = 2$	**Cover**
$g = 0$	$n = 4$	**Plates 59, 60, 61**
$g = 0$	$n = 5$	**Plates 62, 63**
$g = 1$	$n = 5$	**Plate 64**

```
program boy

call parametres

call surface
call intersection

call pssexit
stop
end
```

```
subroutine parametres

include 'boycom.inc'

ne = questionR('$how many ellipses do you want to draw :@')
np = questionR('$how many points in each ellipse ..... :@')
g  = questionR('$give the value of g (0 to 1) ........ :@')
n  = questionI('$give the value of n (n>=2 sym.order)  :@')
npi= questionR('$how many points in the intersection . :@')

return
end
```

```
subroutine surface

include 'boycom.inc'

logical pl

pi=3.1415927
r2=sqrt(2.)

call newgraph

do i=0,2*ne-1              ! for each ellipse
     e=float(i)
     pl=.false.
     do j=0,np             ! for each point of the ellipse
          h=float(j)

          a=h*pi/np-pi/2
          b=e*pi/ne

          c=cos(a)/(1-(g*sin(2*a)*sin(n*b))/r2)

          x=c*((r2/n)*cos(a)*cos((n-1)*b)+(n-1)*sin(a)*cos(b)/n)
          y=c*((r2/n)*cos(a)*sin((n-1)*b)-(n-1)*sin(a)*sin(b)/n)
          z=c*cos(a)-2./3.

          call graph(pl,x,y,z)
          pl=.true.
      end do
end do

call endgraph

return
end
```

```
subroutine intersection

include 'boycom.inc'

s2=sqrt(2.)

if (n.eq.2)                               call interNeq2

if (n.eq.3.and.g.eq.0)                    call interNeq3Geq

if (n.eq.3.and.g.gt.0.and.g.lt.(s2-1)**2) call interNeq3Glt

if (n.eq.3.and.g.gt.0.and.g.ge.(s2-1)**2) call interNeq3Gge

return
end




subroutine interNeq2     ! intersection N=2

include 'boycom.inc'

logical pl

call newgraph

pl=.false.
do i=0,npi
        ri=float(i)

        t=-0.5*atan(4./3.) + (ri/(2*npi))*(atan(2.)+atan(4./3.))

        c2t=cos(2*t)
        s2t=sin(2*t)

        r=5*(3*c2t-4*s2t)* sqrt( abs((3*s2t+4*c2t)*(2*c2t-s2t)) )

        x0=  3.*s2t*s2t*s2t -288.*s2t*s2t*c2t
1          +91.*s2t*c2t*c2t - 54.*c2t*c2t*c2t
        x1=r*cos(t)
        x2=r*sin(t)
        x3=25*(s2t-2*c2t)

        x=x1/x0
        y=x2/x0
        z=x3/x0-2./3.

        call graph(pl,X,Y,Z)
        pl=.true.
end do
```

```
pl=.false.
do i=0,npi
       ri=float(i)

       t=-0.5*atan(4./3.) + (ri/(2*npi))*(atan(2.)+atan(4./3.))

       c2t=cos(2*t)
       s2t=sin(2*t)

       r=5*(3*c2t-4*s2t)* sqrt( abs((3*s2t+4*c2t)*(2*c2t-s2t)) )

       x0=  3.*s2t*s2t*s2t -288.*s2t*s2t*c2t
1         +91.*s2t*c2t*c2t - 54.*c2t*c2t*c2t
       x1=r*cos(t)
       x2=r*sin(t)
       x3=25*(s2t-2*c2t)

       x=-x2/x0
       y= x1/x0
       z= x3/x0-2./3.

       call graph(pl,X,Y,Z)
       pl=.true.
end do

pl=.false.
do i=0,npi




       ri=float(i)

       t=-0.5*atan(4./3.) + (ri/(2*npi))*(atan(2.)+atan(4./3.))

       c2t=cos(2*t)
       s2t=sin(2*t)

       r=5*(3*c2t-4*s2t)* sqrt( abs((3*s2t+4*c2t)*(2*c2t-s2t)) )

       x0=  3.*s2t*s2t*s2t -288.*s2t*s2t*c2t
1         +91.*s2t*c2t*c2t - 54.*c2t*c2t*c2t
       x1=r*cos(t)
       x2=r*sin(t)
       x3=25*(s2t-2*c2t)

       x= x2/x0
       y=-x1/x0
       z= x3/x0-2./3.

       call graph(pl,X,Y,Z)
       pl=.true.
end do
```

```
pl=.false.
do i=0,npi
        ri=float(i)

        t=-0.5*atan(4./3.) + (ri/(2*npi))*(atan(2.)+atan(4./3.))

        c2t=cos(2*t)
        s2t=sin(2*t)

        r=5*(3*c2t-4*s2t)* sqrt( abs((3*s2t+4*c2t)*(2*c2t-s2t)) )

        x0=  3.*s2t*s2t*s2t -288.*s2t*s2t*c2t
1          +91.*s2t*c2t*c2t - 54.*c2t*c2t*c2t
        x1=r*cos(t)
        x2=r*sin(t)
        x3=25*(s2t-2*c2t)

        x=-x1/x0
        y=-x2/x0
        z= x3/x0-2./3.

        call graph(pl,X,Y,Z)
        pl=.true.
end do

call endgraph

return
end



        subroutine interNeq3Geq ! intersection  N=3 G=0

        include 'boycom.inc'

        R2=SQRT(2.)
        R3=SQRT(3.)

        call newgraph

C line 1
        X1=20.*R2/3.
        X2=0.
        X3=-6.-2./3.
        call graph(.false.,X1,X2,X3)

        X1=-20.*R2/3.
        X2=0.
        X3=22./3.-2./3.
        call graph(.true.,X1,X2,X3)

C line 2
        X1=-10.*R2/3.
        X2= 10.*R2/R3
        X3=-6.-2./3.
        call graph(.false.,X1,X2,X3)
```

```
        X1=10*R2/3.
        X2=-10*R2/R3
        X3=22./3.-2./3.
        call graph(.true.,X1,X2,X3)

C line 3
        X1=-10.*R2/3.
        X2=-10.*R2/R3
        X3=-6.-2./3.
        call graph(.false.,X1,X2,X3)

        X1=10.*R2/3.
        X2=10.*R2/R3
        X3=22./3.-2./3.
        call graph(.true.,X1,X2,X3)

        pl=.false.
        do i=1,npi-1
                t=-pi/6. + pi/3.*float(i)/npi
                uscos=1/(3*cos(3*t))
                x=uscos*2*s2*cos(t)
                y=uscos*2*s2*sin(t)
                z=-2./3.
                call graph(pl,x,y,z)
                pl=.true.
        end do

        pl=.false.
        do i=1,npi-1
                t=pi/6. + pi/3.*float(i)/npi
                uscos=1/(3*cos(3*t))
                x=uscos*2*s2*cos(t)
                y=uscos*2*s2*sin(t)
                z=-2./3.




                call graph(pl,x,y,z)
                pl=.true.
        end do

        pl=.false.
        do i=1,npi-1
                t=pi/2. + pi/3.*float(i)/npi
                uscos=1/(3*cos(3*t))
                x=uscos*2*s2*cos(t)
                y=uscos*2*s2*sin(t)
                z=-2./3.
                call graph(pl,x,y,z)
                pl=.true.
        end do

        call endgraph

        return
        end
```

```
subroutine interNeq3Gge         ! intersection n=3 g>(sqrt(2)-1)**2

include 'boycom.inc'

logical pl

pi=3.1415927
r2=sqrt(2.)

call newgraph

a=atan(g)

pl=.false.
do i=0,npi
        t=(pi*float(i))/npi
        apt=a+t

        ss=3*(3*sin(2*a)+sin(6*apt))
        if ( abs(ss).le.0.01 ) then
                                   x=0.
                                   y=0.
                                   z=0.
                                   goto 9
                                 end if

        r=(4*r2*cos(a)*sin(3*apt))/ss

        x=r*cos(t)
        y=r*sin(t)
        z=6*sin(2*a)/ss-2./3.

        call graph(pl,x,y,z)
        pl=.true.
end do

call endgraph

return
end
```

```
subroutine interNeq3Glt          ! intersection n=3 0<g<(sqrt(2)-1)**:

include 'boycom.inc'

logical pl
real            aa(5000),bb(5000),cc(5000)

pi=3.1415927
r2=sqrt(2.)
r3=sqrt(3.)

call newgraph

a=atan(g)

t1=-asin(3*sin(2*a))/6.-a
t2= asin(3*sin(2*a))/6.-a+pi/6.
t3=-asin(3*sin(2*a))/6.-a+pi/3.
t4=t1
t5=t2

do ir=1,2

        do ih=1,npi+1
                h=float(ih)
                xh=-1+2*(h)/(npi+2.)
                t=t4+(t5-t4)*(0.5+xh/(1.+xh*xh))
                u=3*(3*sin(2*a)+sin(6*(a+t)))
                v=4*r2*cos(a)*cos(t)*sin(3*(a+t))
                w=4*r2*cos(a)*sin(t)*sin(3*(a+t))
                aa(ih)=v/u
                bb(ih)=w/u
                cc(ih)=6.*sin(2*a)/u
        end do

        xx1=-20.*r2/3.
        yy1=0.
        xx2=-10.*r2/3.
        yy2=-10.*r2/r3

        do is=0,2
                pl=.false.
                if (ir.eq.1.and.g.le.0.01) then
                                x=xx1
                                y=yy1
                                z=22./3.-2./3.
                                call graph(.false.,x,y,z)
                                pl=.true.
                        end if
                do ih=1,npi
                        x=aa(ih)
                        y=bb(ih)
                        z=cc(ih)-2./3.
                        call graph(pl,x,y,z)
                        pl=.true.
                end do
                if (ir.eq.2.and.g.le.0.01) then
                                x=xx2
                                y=yy2
                                z=-6.-2./3.
```

```
                                    call graph(.true.,x,y,z)
                              end if

                     if (ir.eq.1.and.g.le.0.01) then
                              x=-(xx1+r3*yy1)/2.
                              y=(-yy1+r3*xx1)/2.
                              xx1=x
                              yy1=y
                              end if
                     if (ir.eq.2.and.g.le.0.01) then
                              x=-(xx2+r3*yy2)/2.
                              y=(-yy2+r3*xx2)/2.
                              xx2=x
                              yy2=y
                              end if

                     do ih=1,npi+1
                              x=-(aa(ih)+r3*bb(ih))/2.
                              y=(-bb(ih)+r3*aa(ih))/2.
                              aa(ih)=x
                              bb(ih)=y
                     end do
          end do

          t4=t2
          t5=t3
end do

call endgraph

return
end




subroutine newgraph
include 'amoi:psamoi.inc'
character*2 cnum
          num=num+1
          write(cnum,'(i2.2)')num
          buf='boyobj'//cnum//':=vec item '
          call pssbuf
return
end

subroutine endgraph
include 'amoi:psamoi.inc'
          call pssvsto(0)
return
end

subroutine graph(pl,x,y,z)
logical pl
include 'amoi:psamoi.inc'
          call pssvsto(4,pl,x,y,z)
return
end
```

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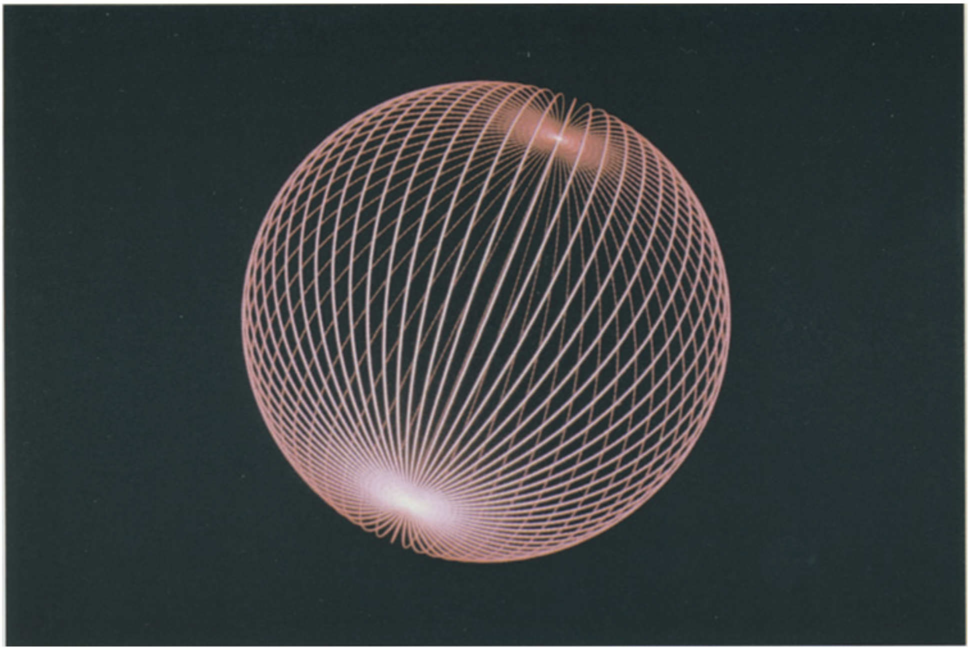
The General Type of Singularity of a Set of $2n-1$ Smooth Functions of n Variables, Duke Journal of Math., Ser. 2, 45, p. 220–293, 1944.

Subject Index

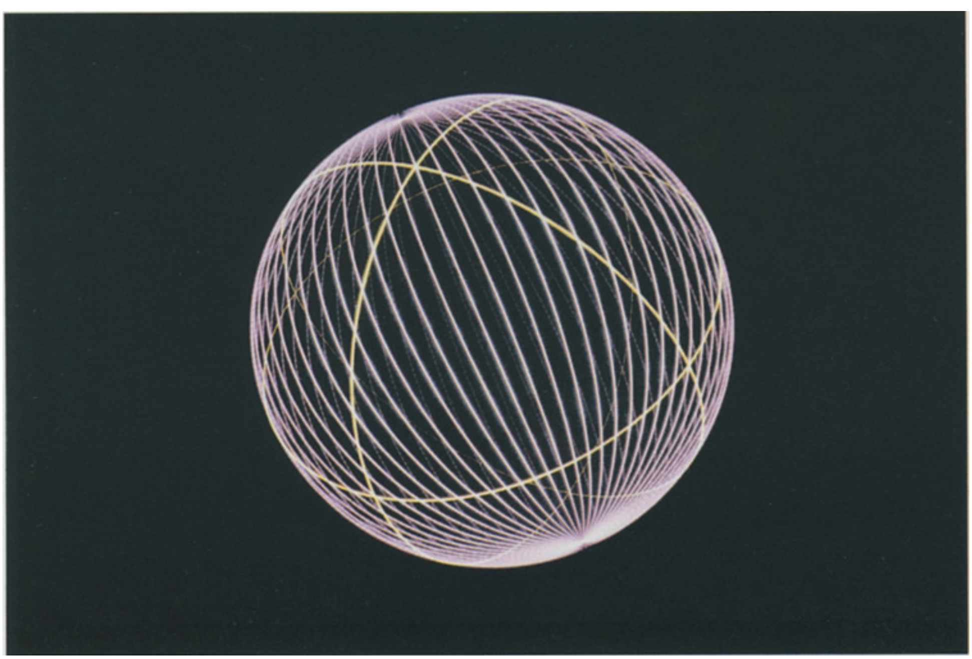
Plate Index

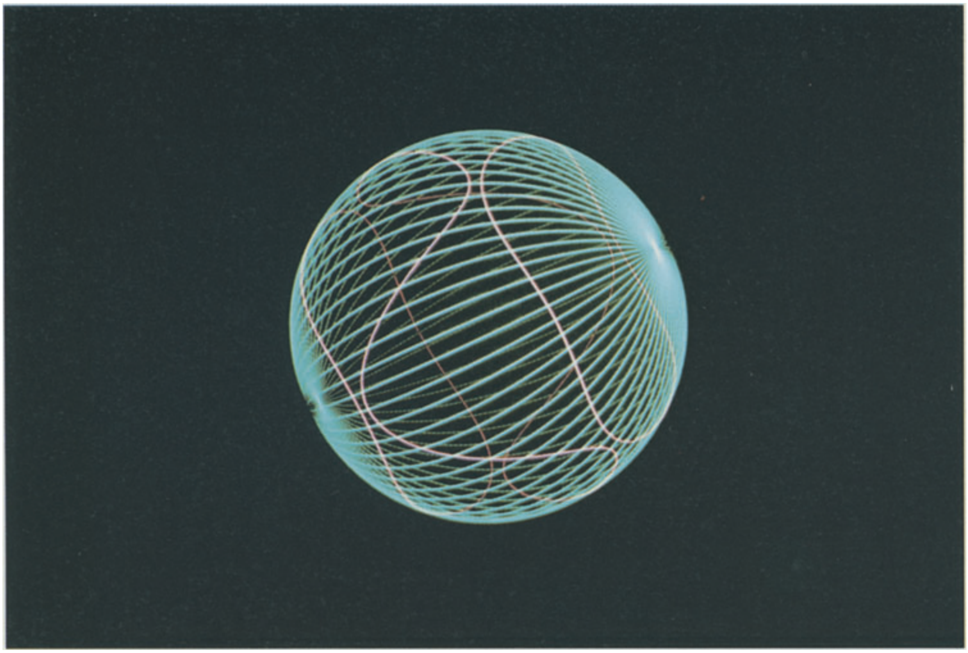
Plates



1 Sphere

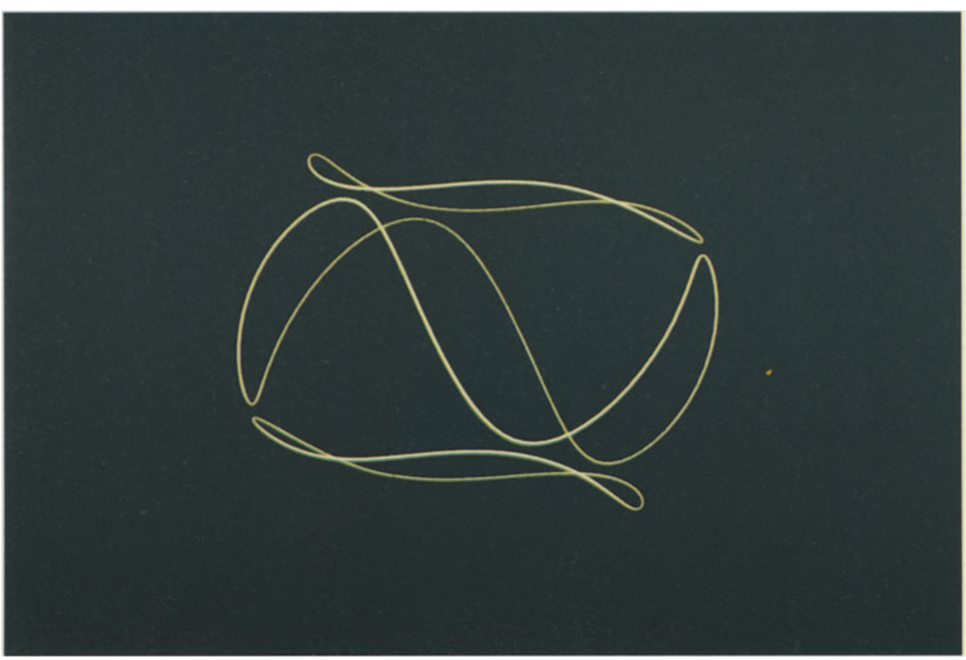
2 Sphere with three great circles

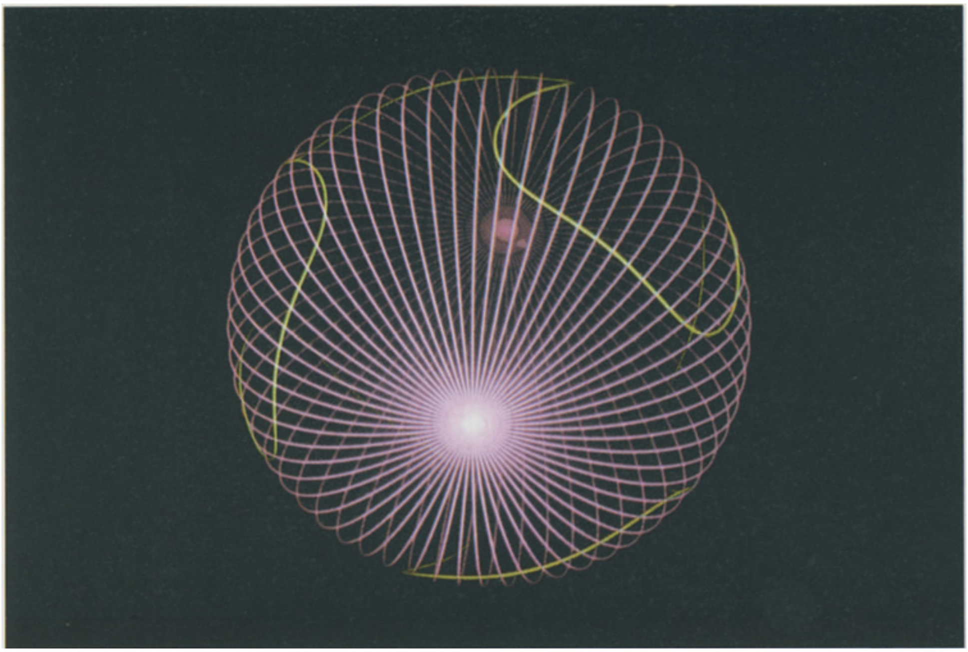




3 Sphere with a nonconnected perturbation of three great circles

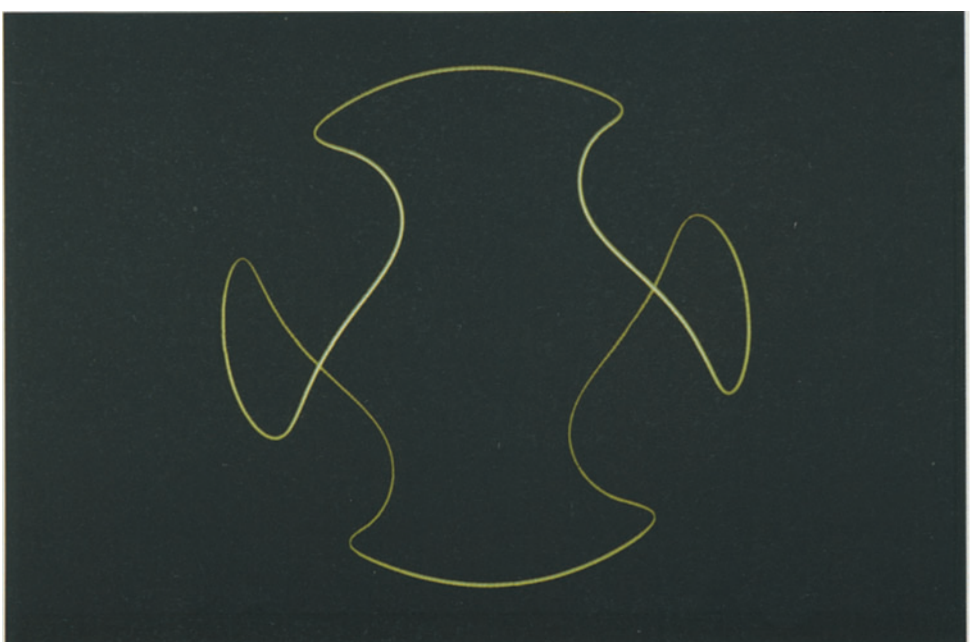
4 Nonconnected perturbation of three great circles

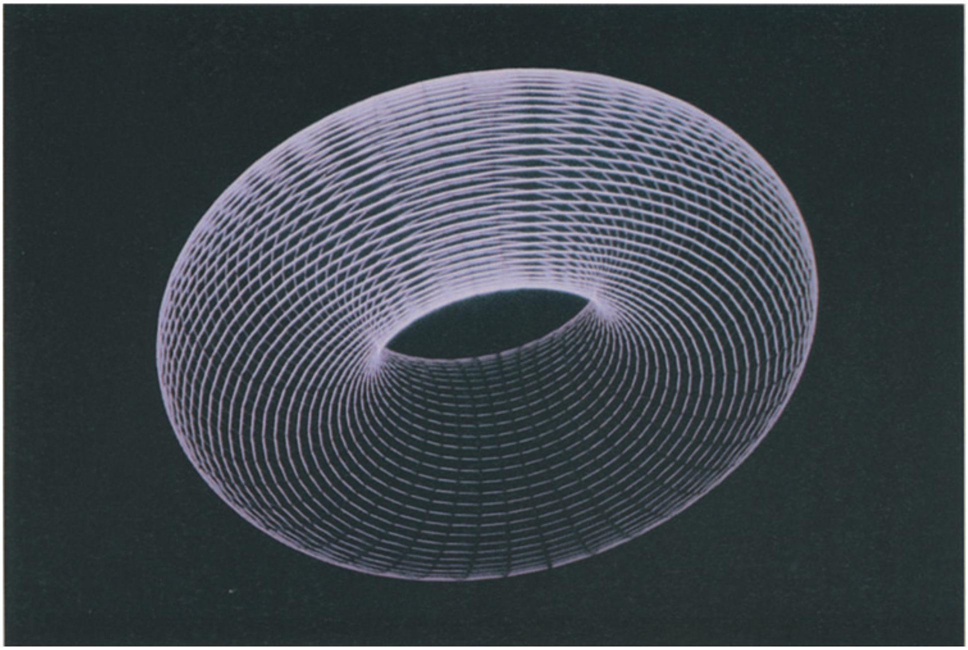




5 Sphere with a connected perturbation of three great circles

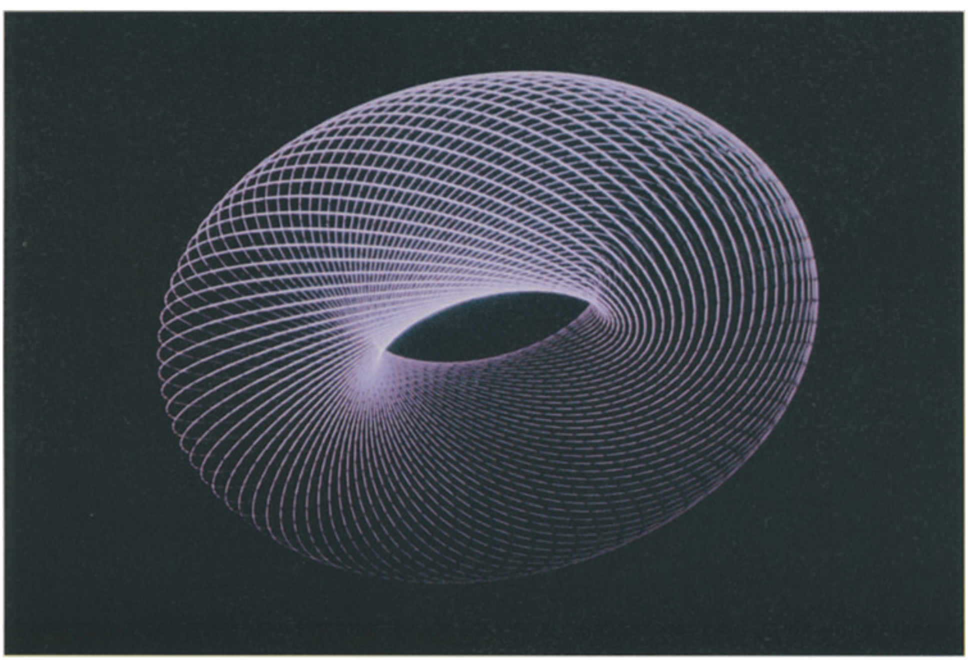
6 Connected perturbation of three great circles

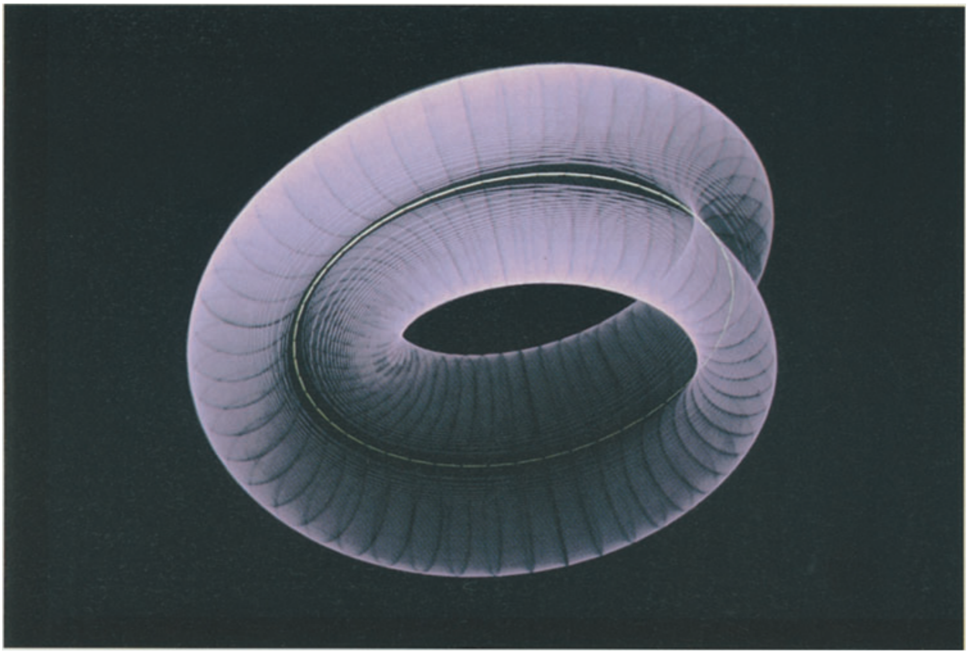




7 Standard torus

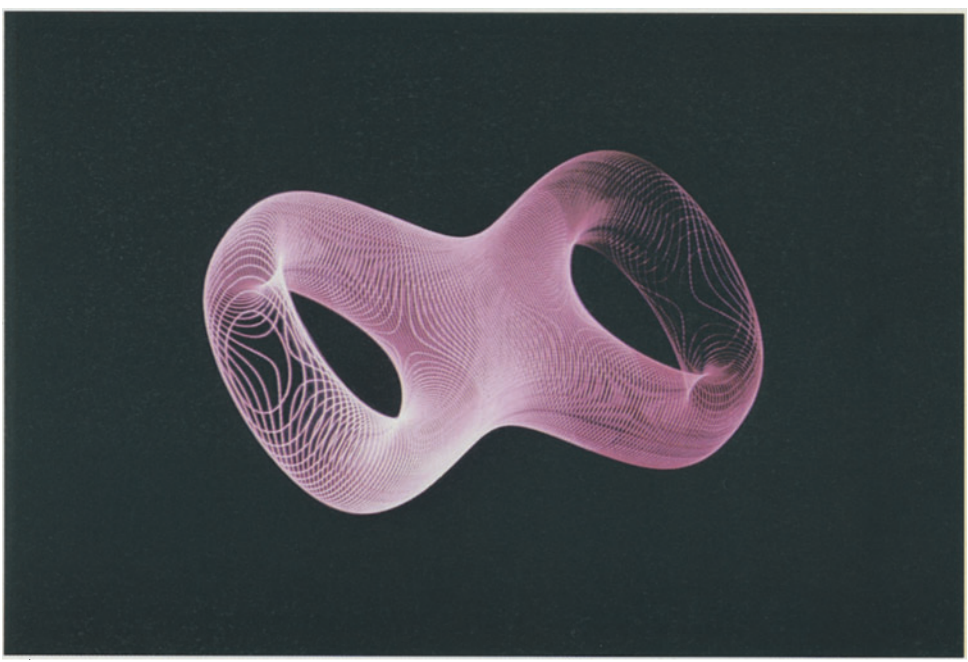
8 Torus generated by Villarceau circles

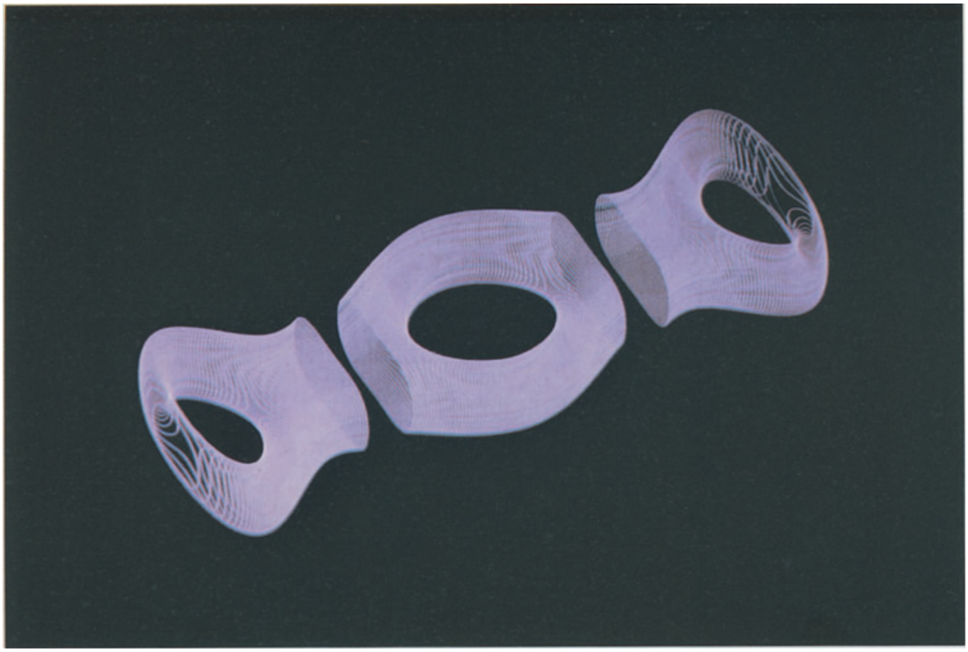




9 Immersed Klein bottle

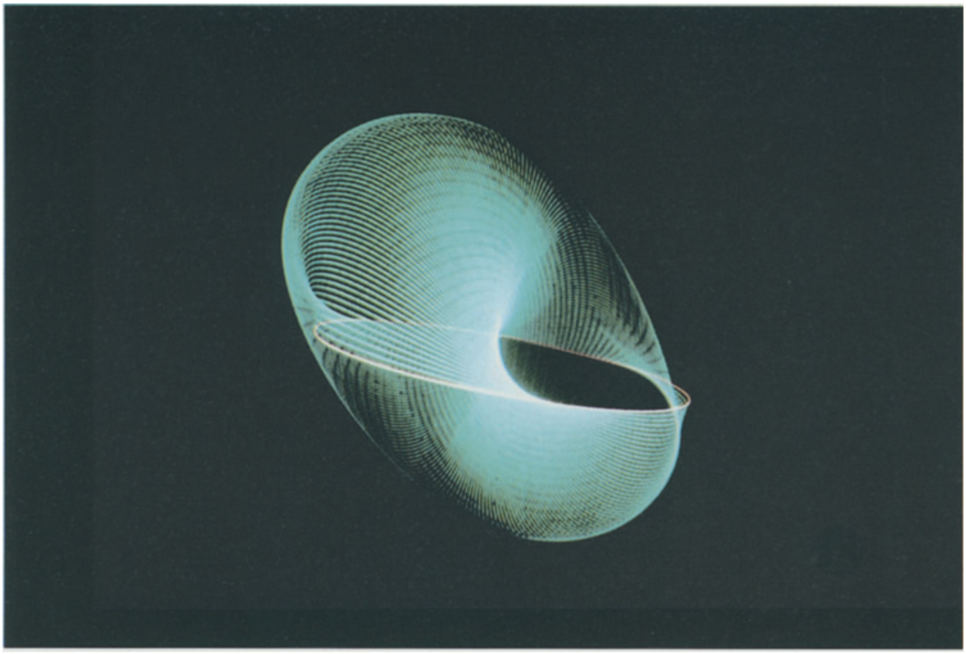
10 Oriented closed surface of genus two





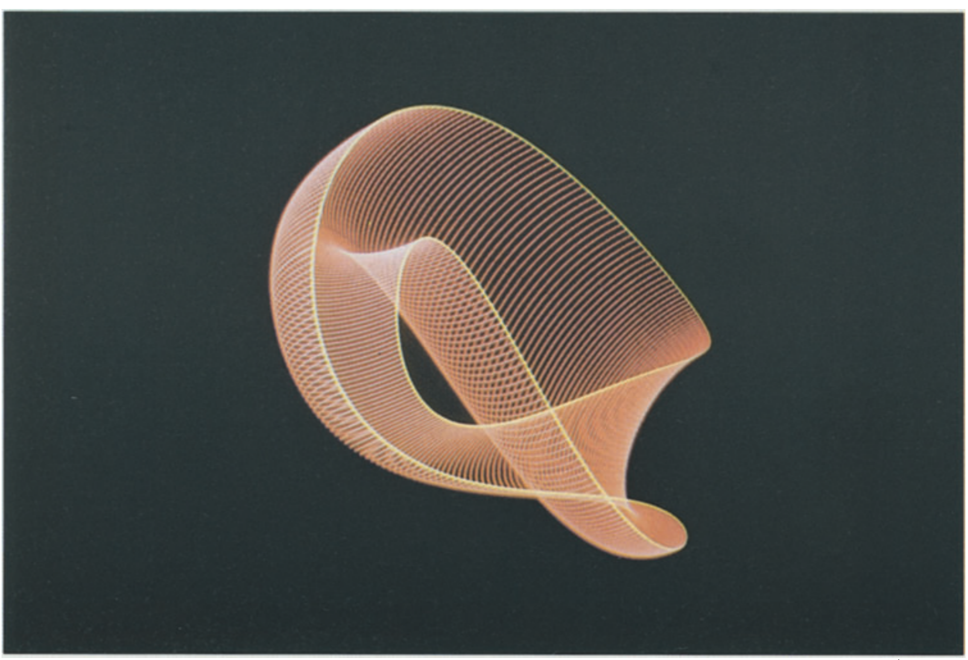
11 Connected sum of three tori

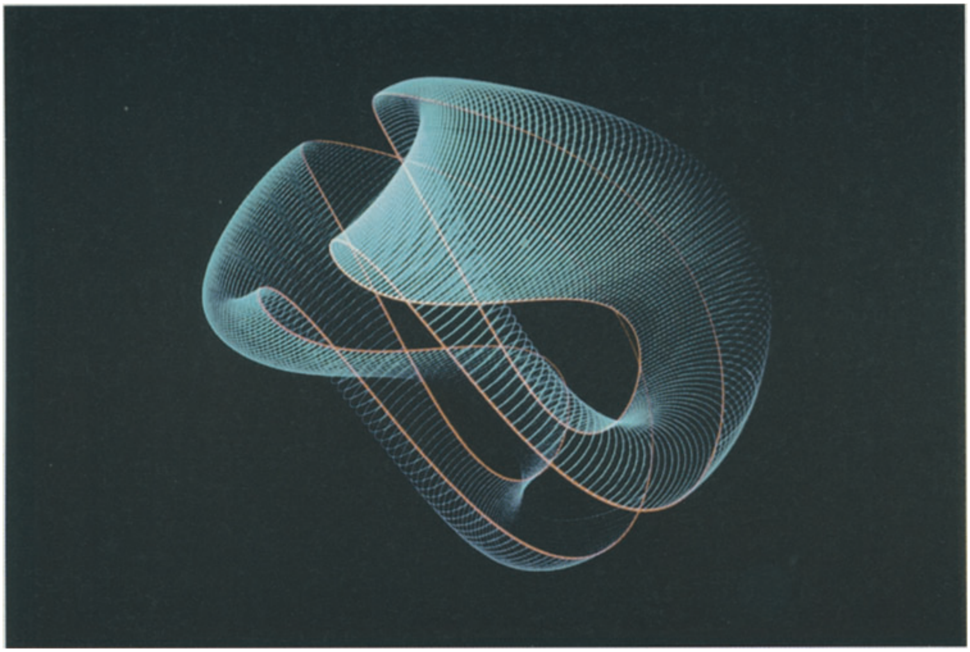
12 Oriented closed surface of genus three



13 Möbius strip with a circle as boundary

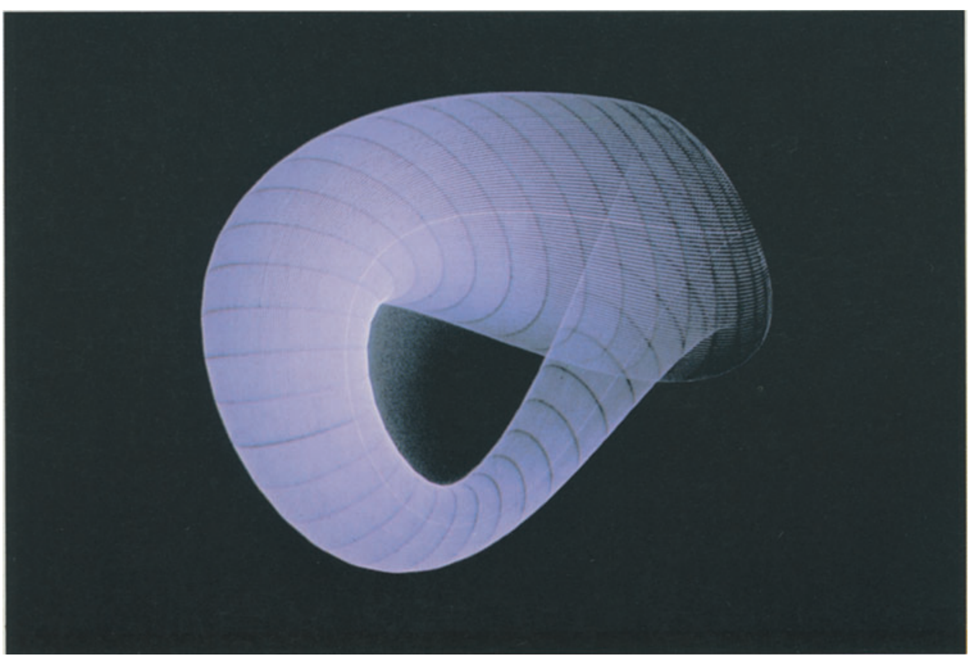
14 Immersed Möbius strip

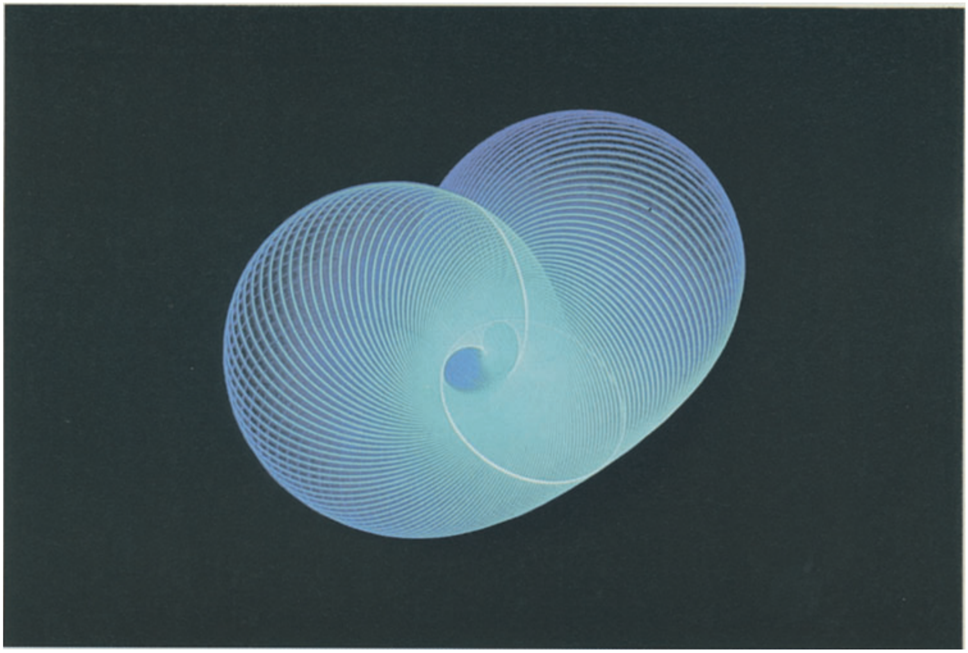




15 Gluing two Möbius strips along their boundaries

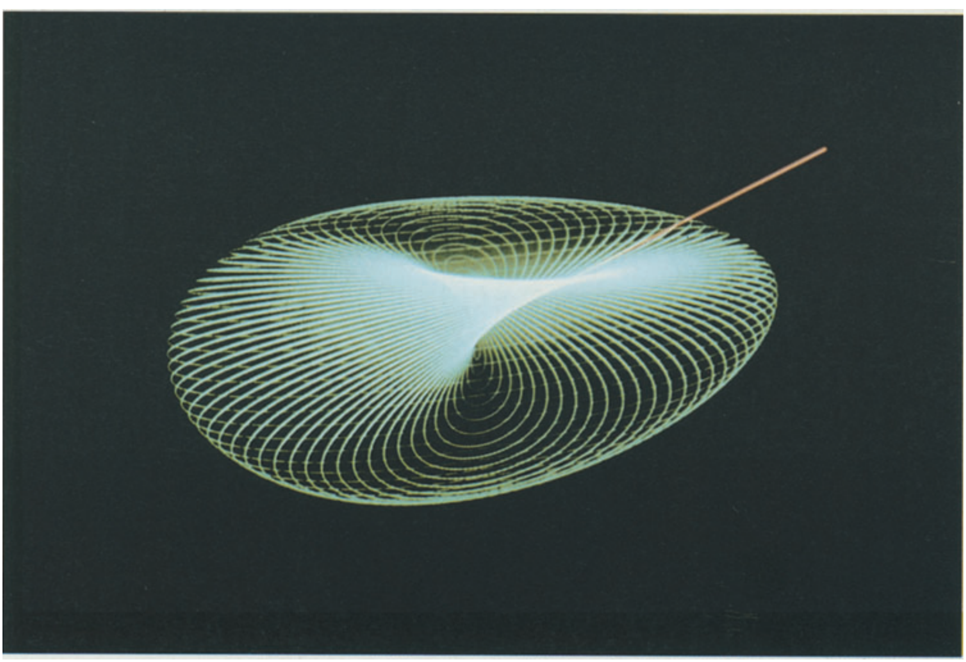
16 Klein bottle

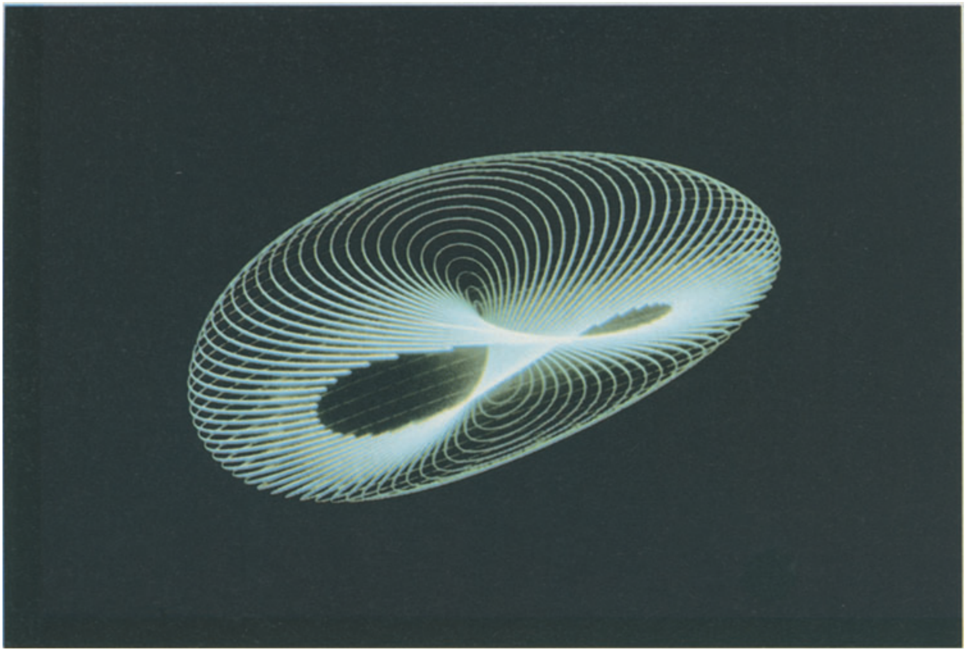




17 Klein bottle obtained by gluing two Möbius strips with circular boundary

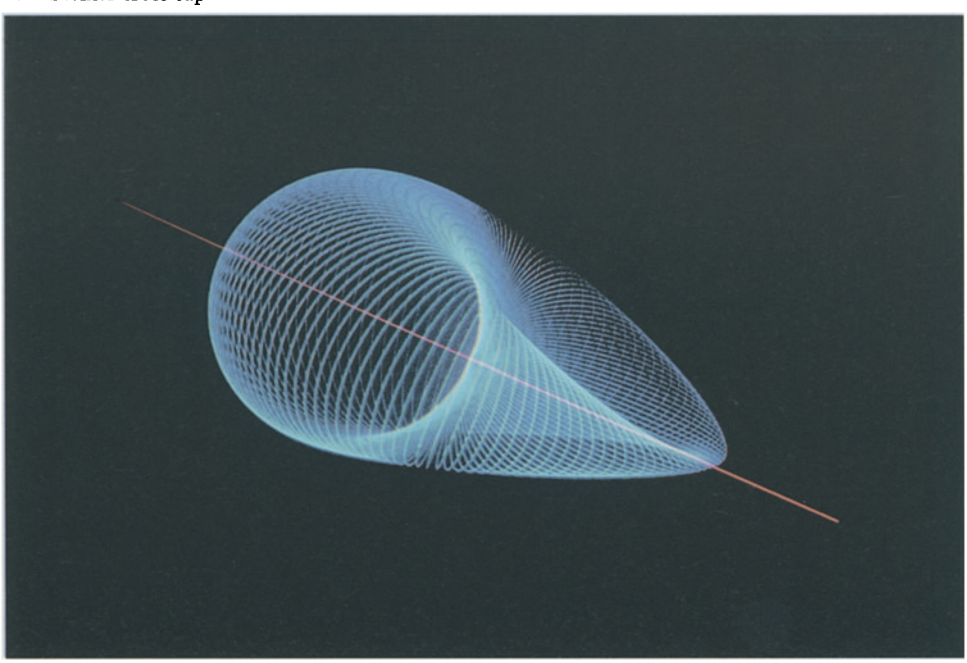
18 Steiner cross-cap

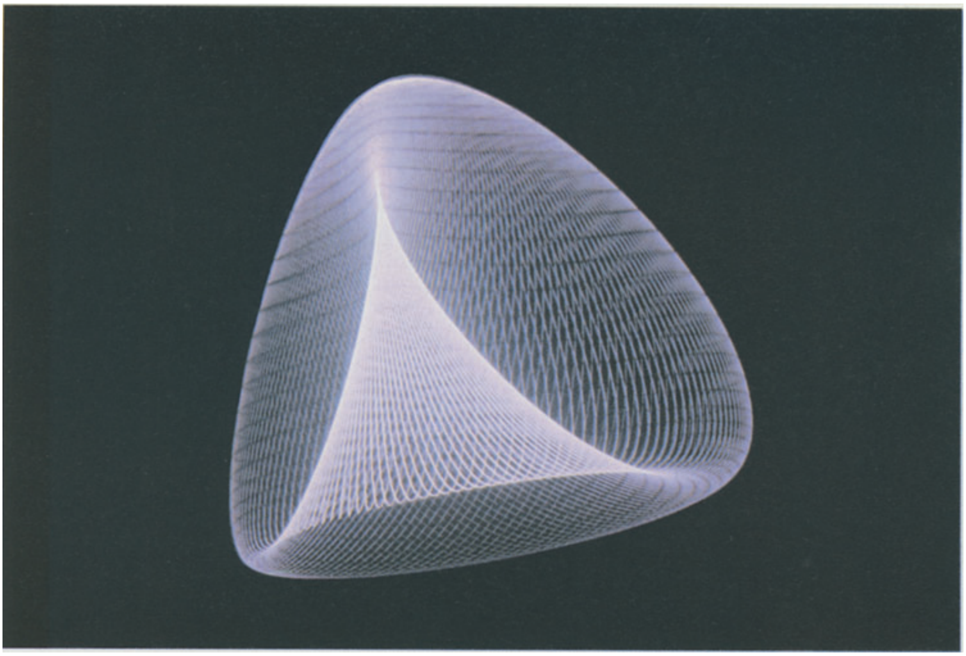




19 Steiner cross-cap with a window

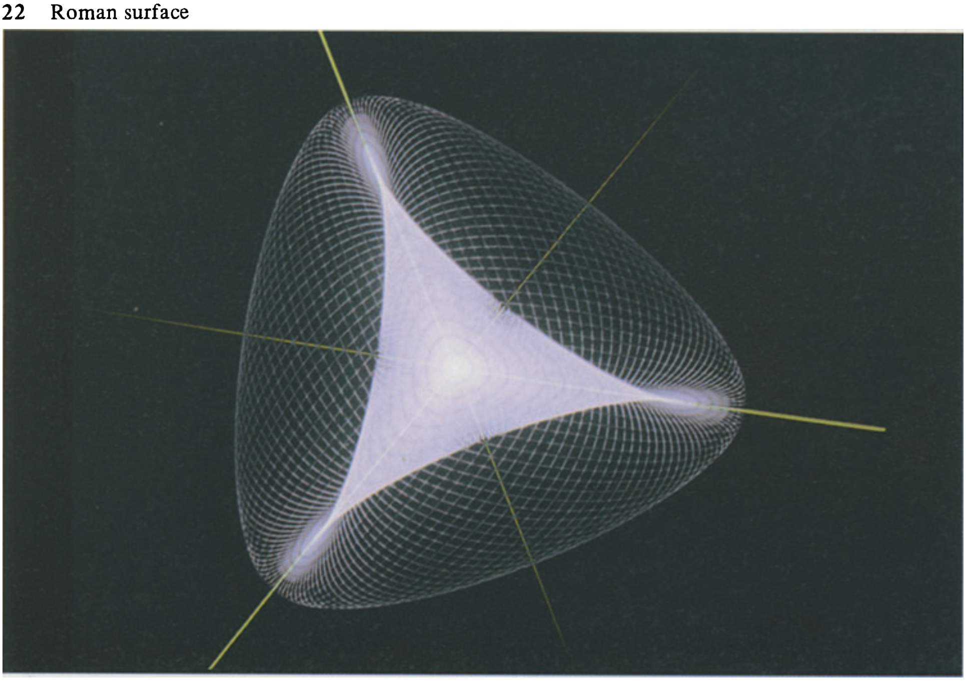
20 Steiner cross-cap

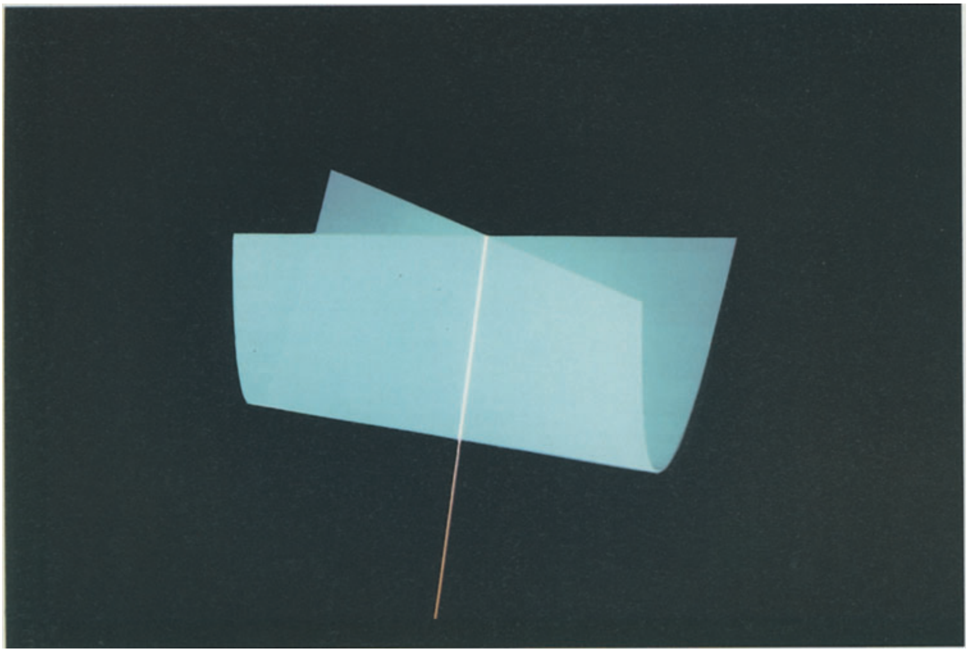




21 Image of the projective plane in the Roman surface

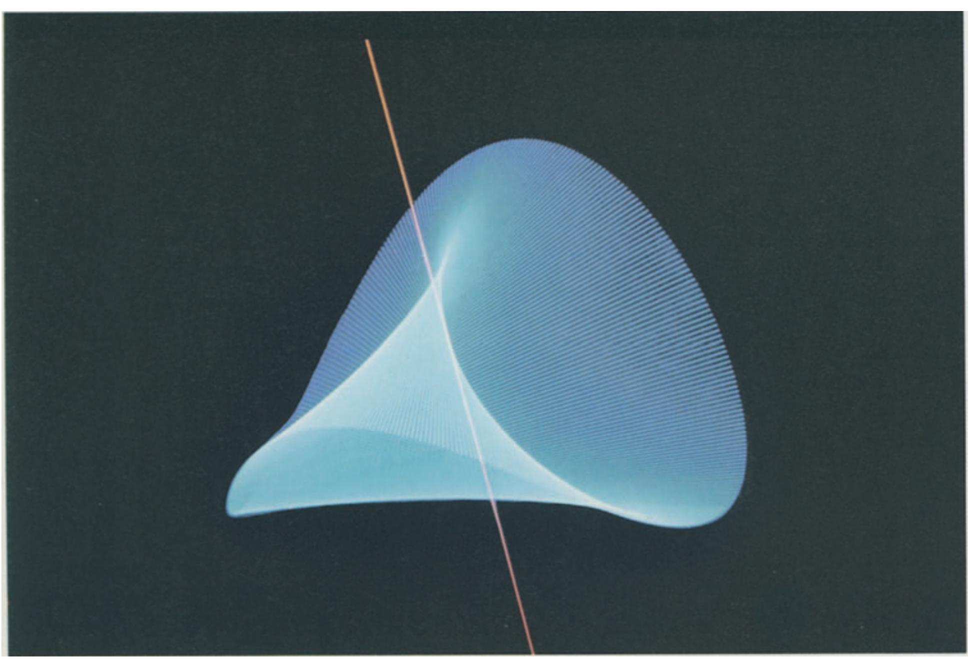
22 Roman surface

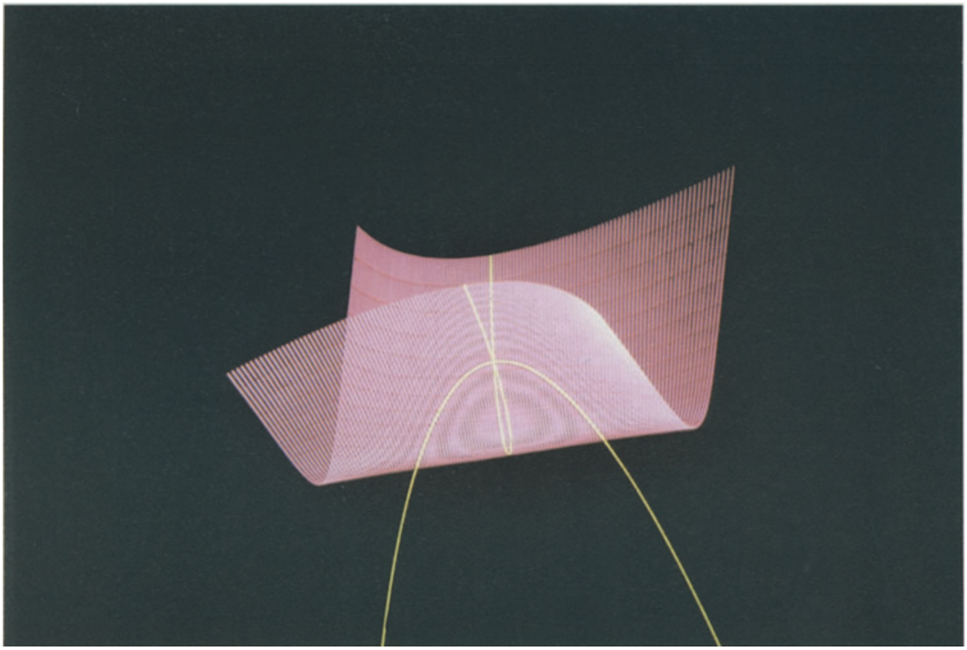




23 Whitney umbrella on the ruled cubic surface

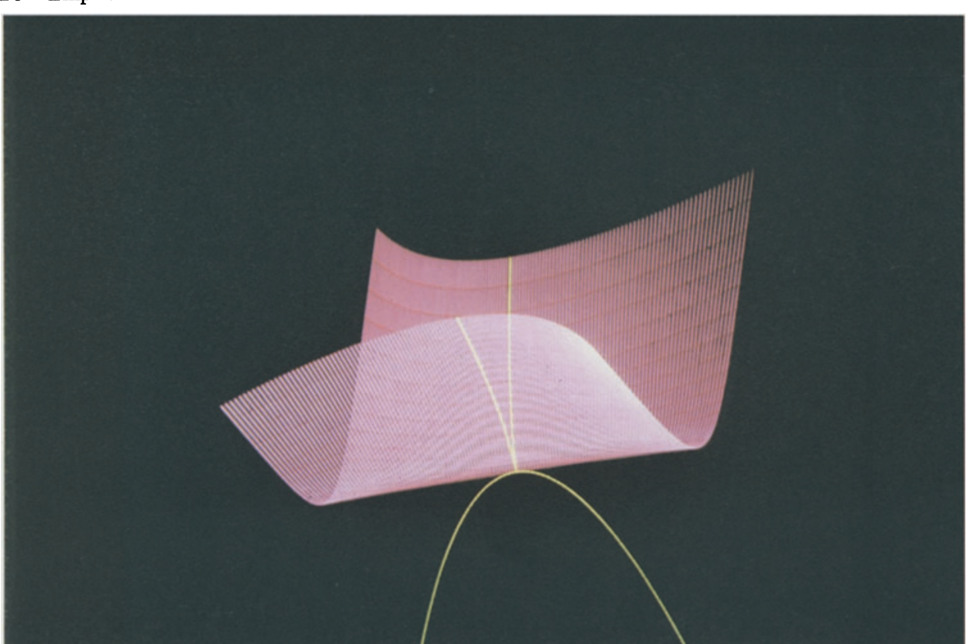
24 Plücker conoid

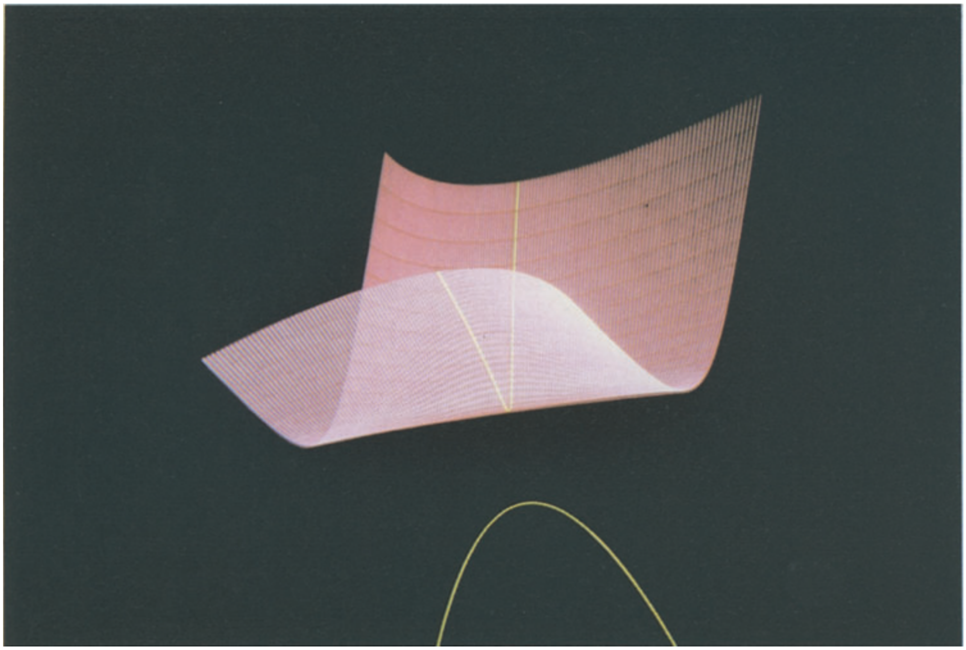




25 Elliptic confluence of two umbrellas: $t > 0$

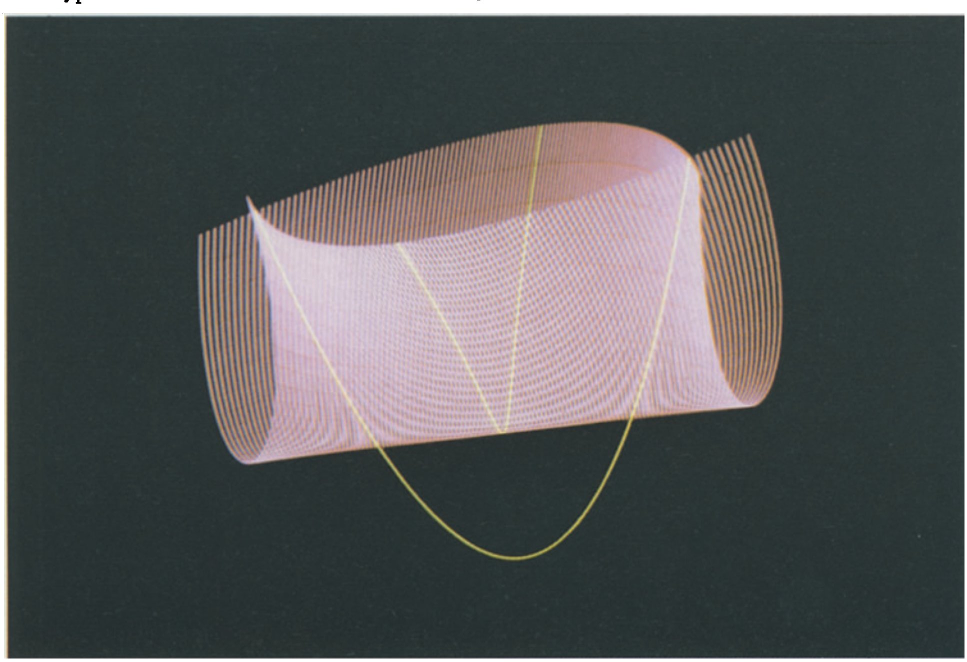
26 Elliptic confluence of two umbrellas: $t = 0$

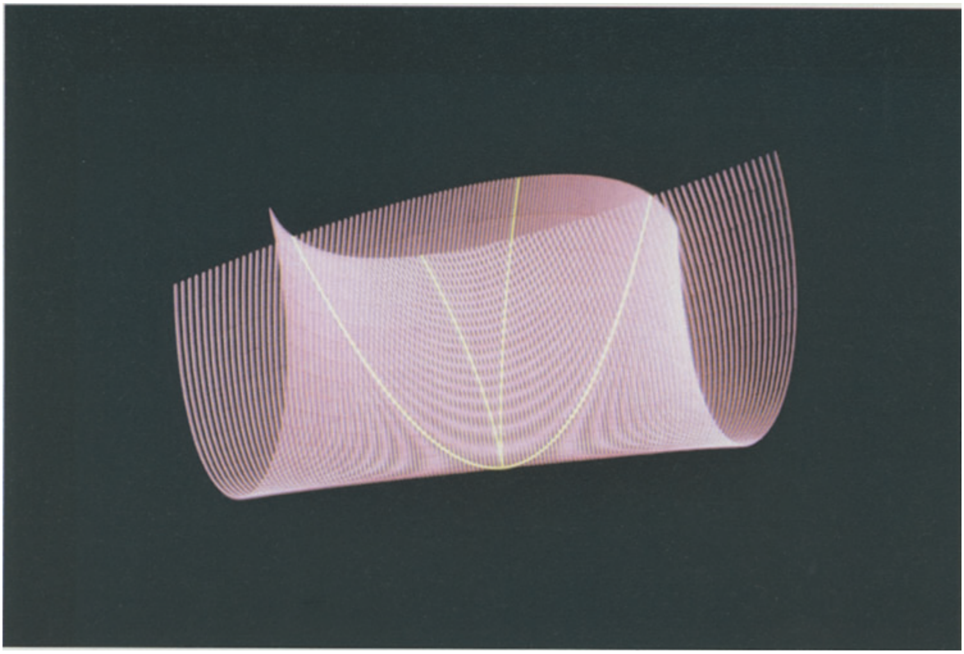




27 Elliptic confluence of two umbrellas: $t < 0$

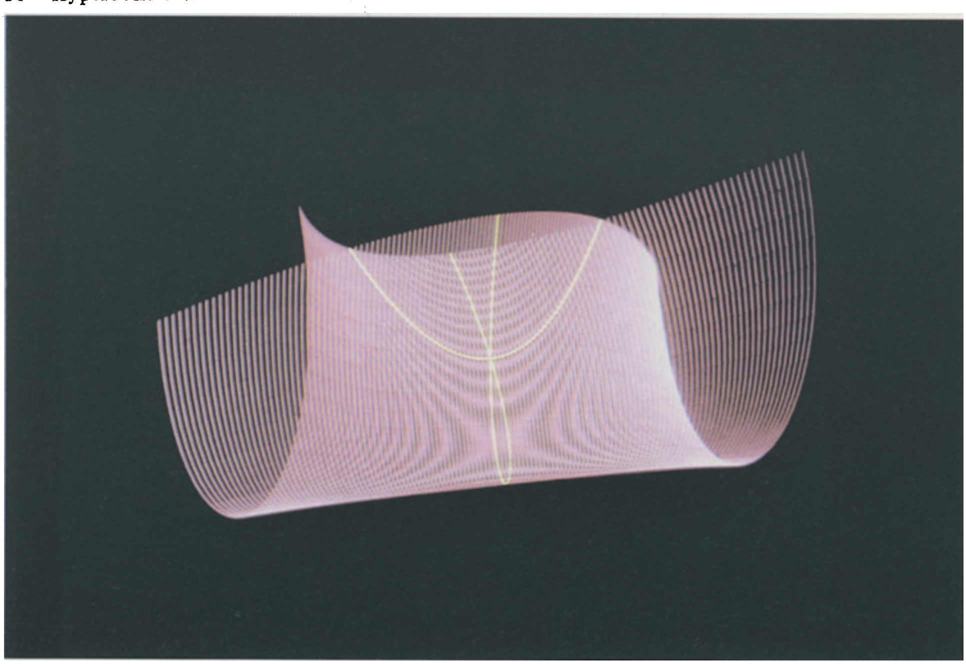
28 Hyperbolic confluence of two umbrellas: $t > 0$

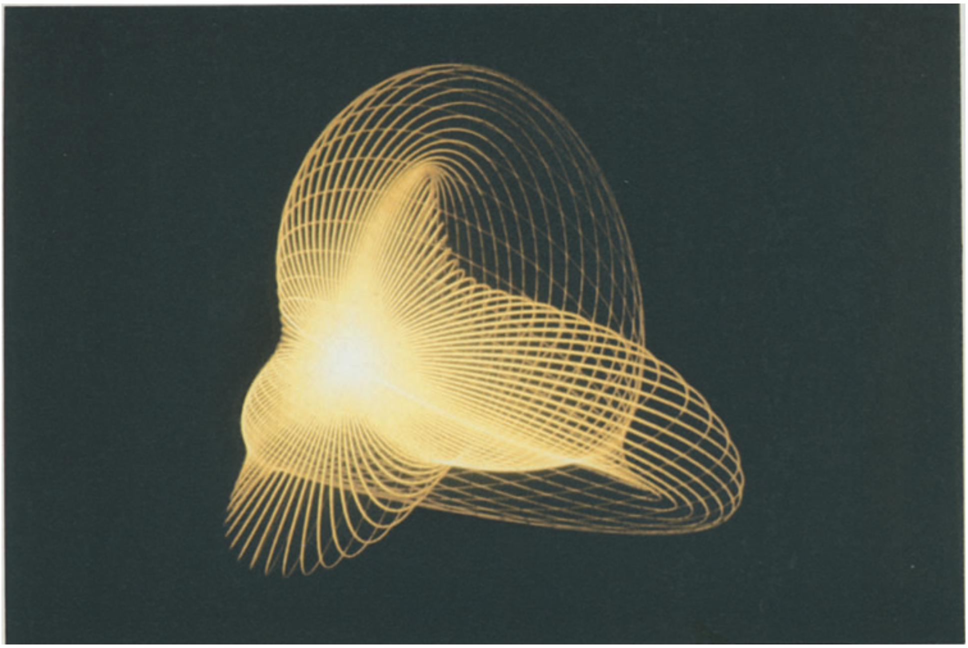




29 Hyperbolic confluence of two umbrellas: t = 0

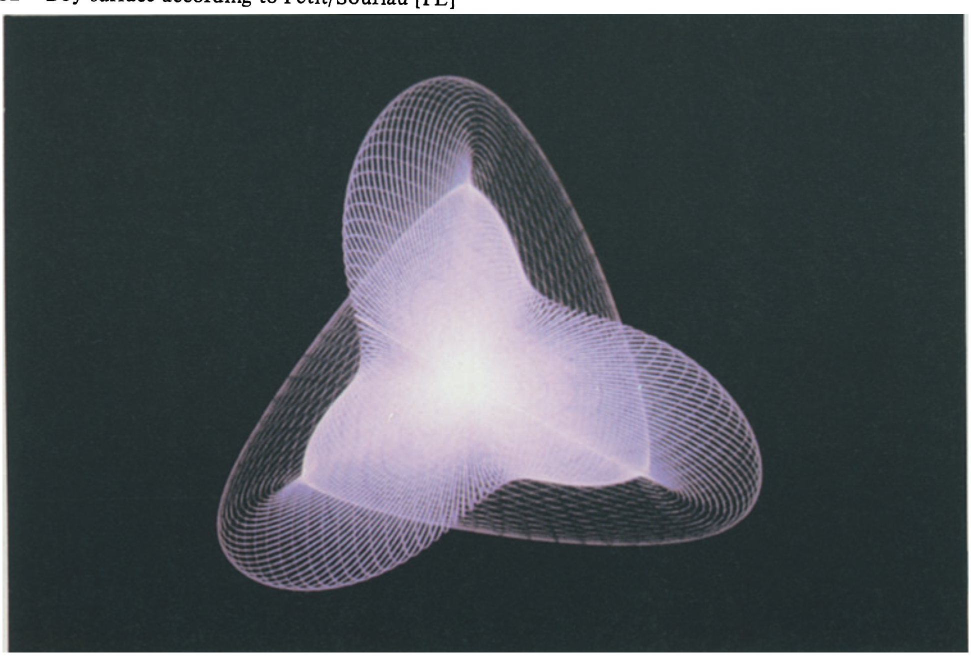
30 Hyperbolic confluence of two umbrellas: $t < 0$

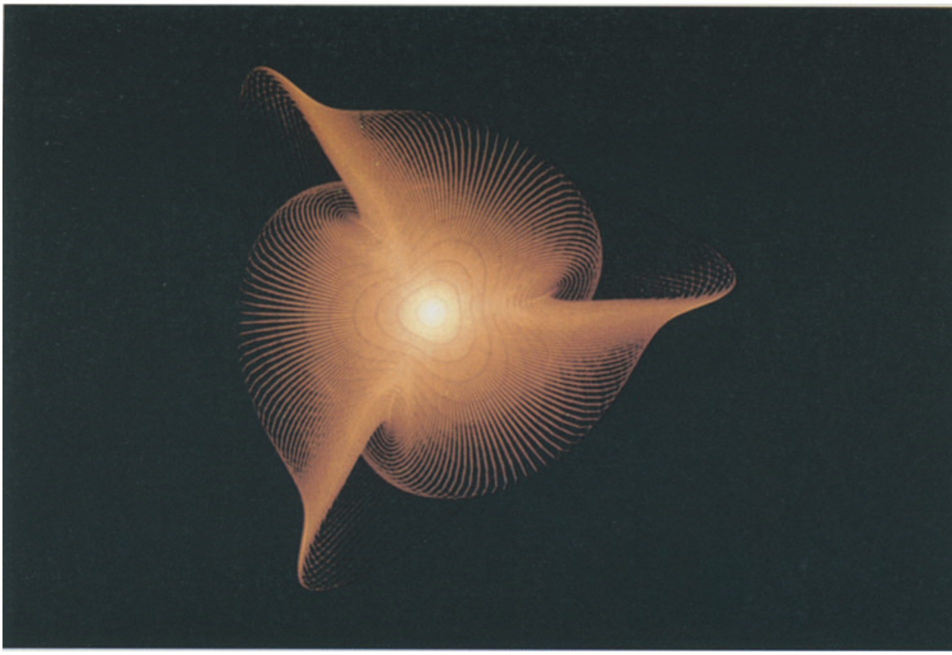




31 Boy surface according to Petit/Souriau [PE]

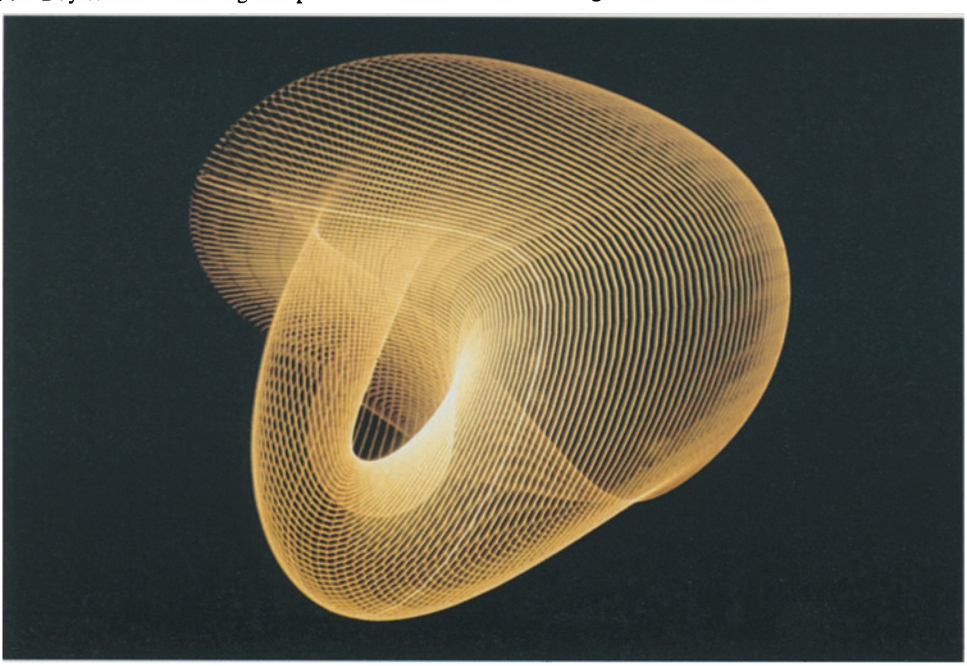
32 Boy surface according to Petit/Souriau [PE]

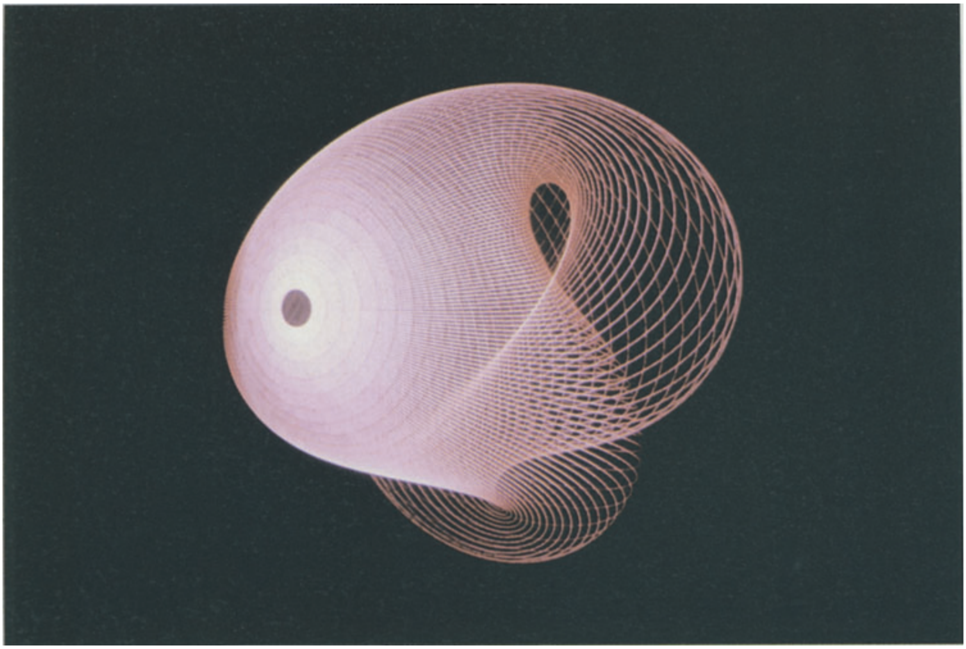




33 Boy surface according to Morin [MO2]

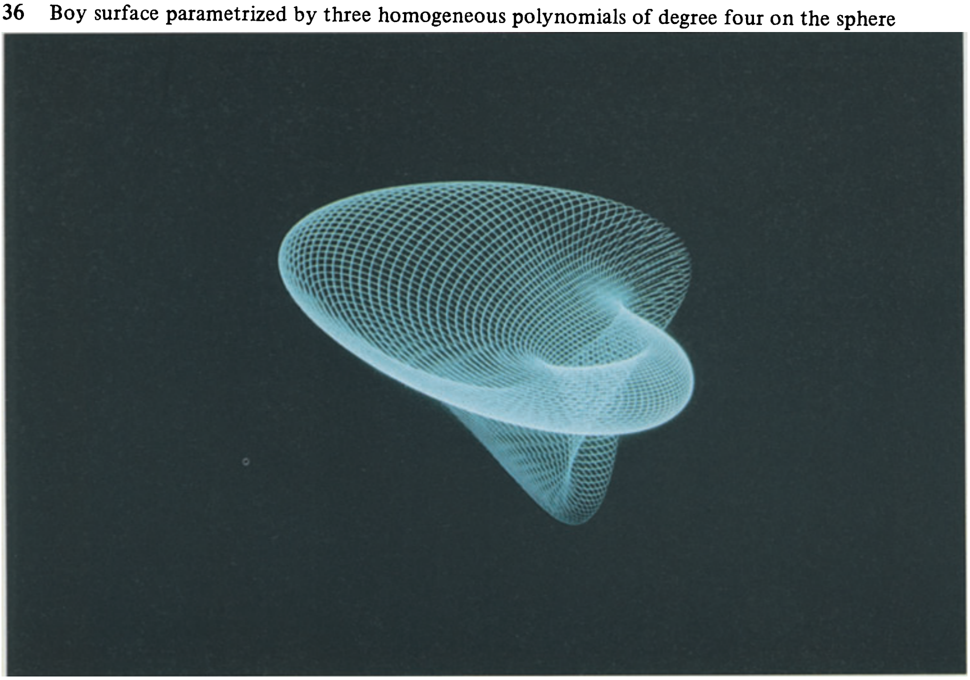
34 Boy surface according to a parametrization due to J. F. Hughes

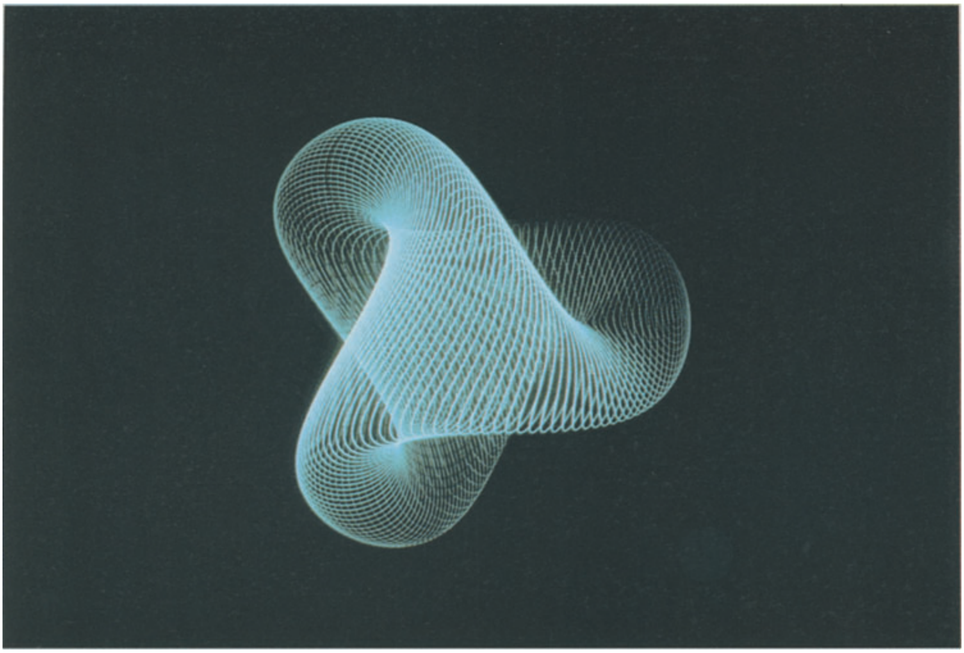




35 Boy surface according to a parametrization due to R. Bryant

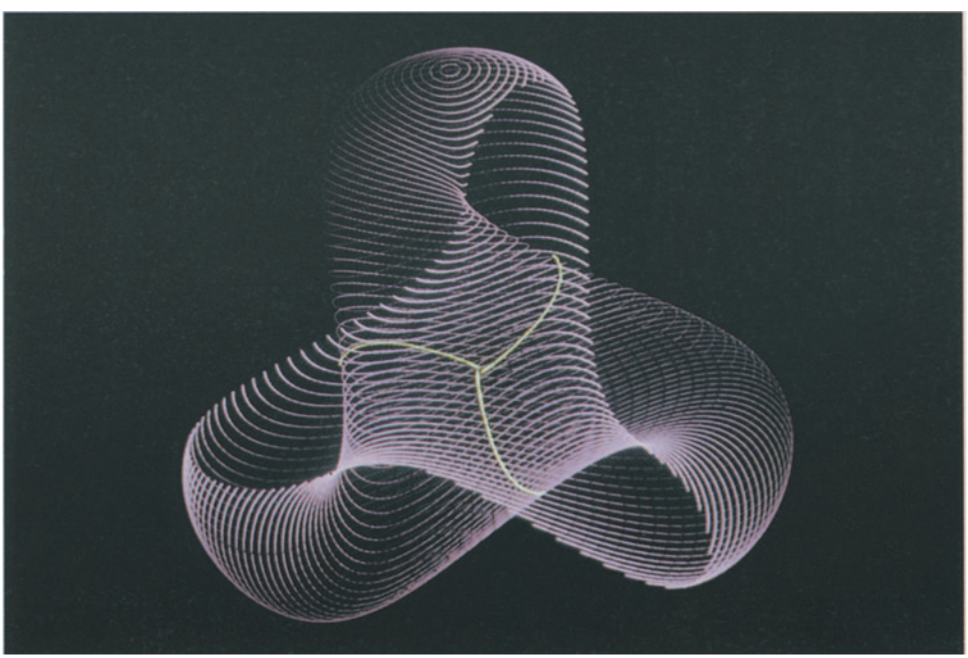
36 Boy surface parametrized by three homogeneous polynomials of degree four on the sphere

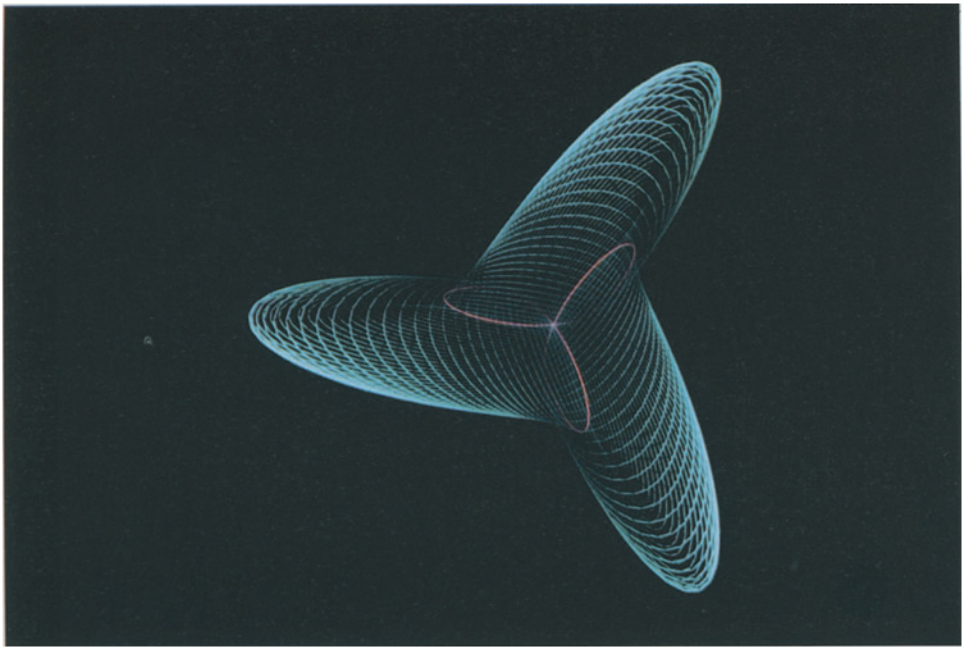




37 Boy surface parametrized by three homogeneous polynomials of degree four on the sphere

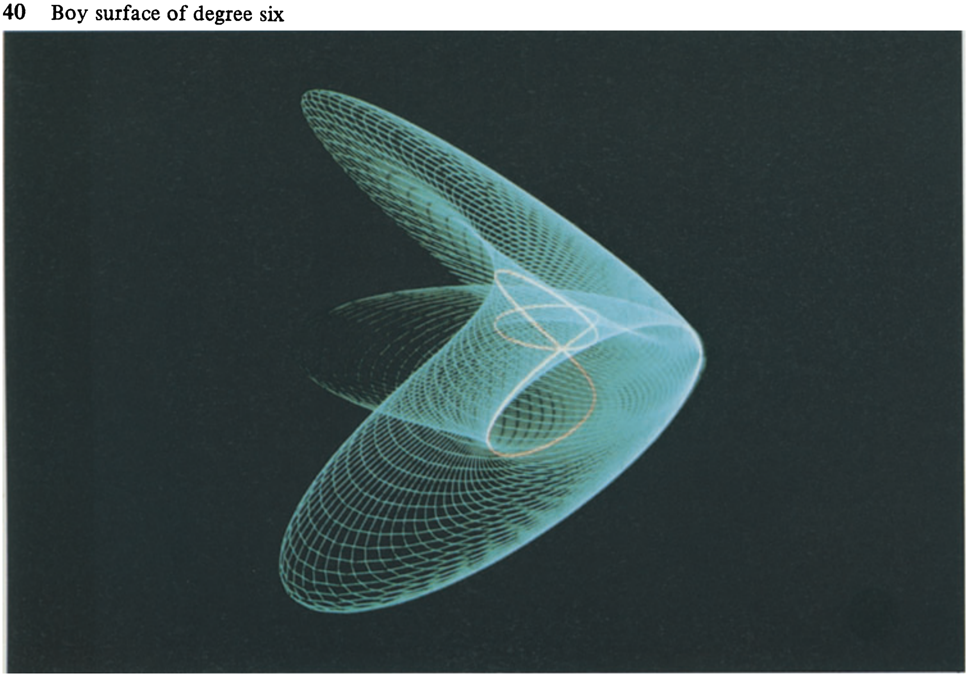
38 Boy surface parametrized by three homogeneous polynomials of degree four on the sphere with a window

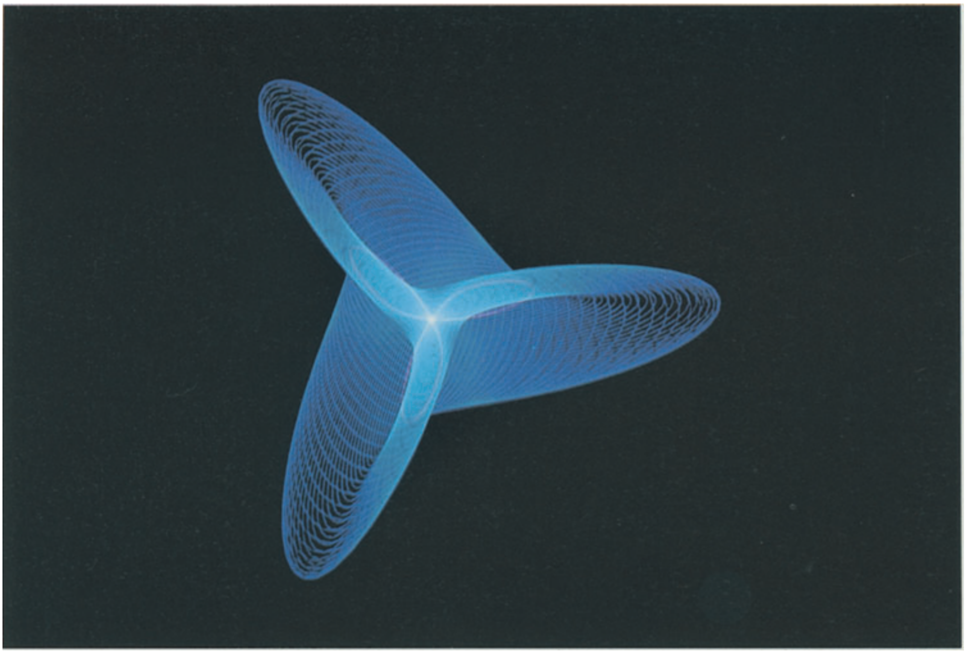




39 Boy surface of degree six

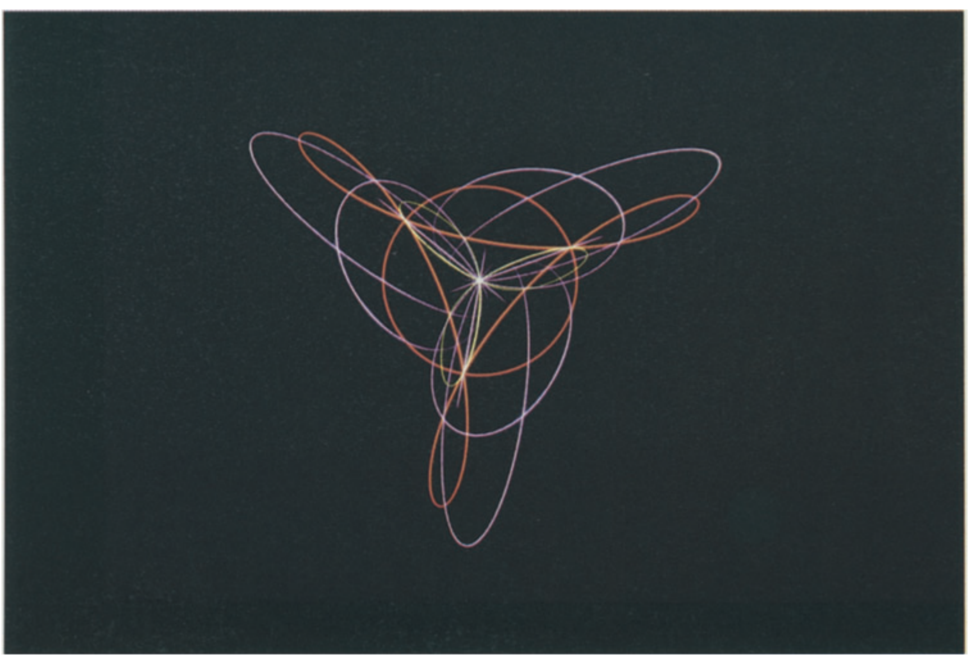
40 Boy surface of degree six

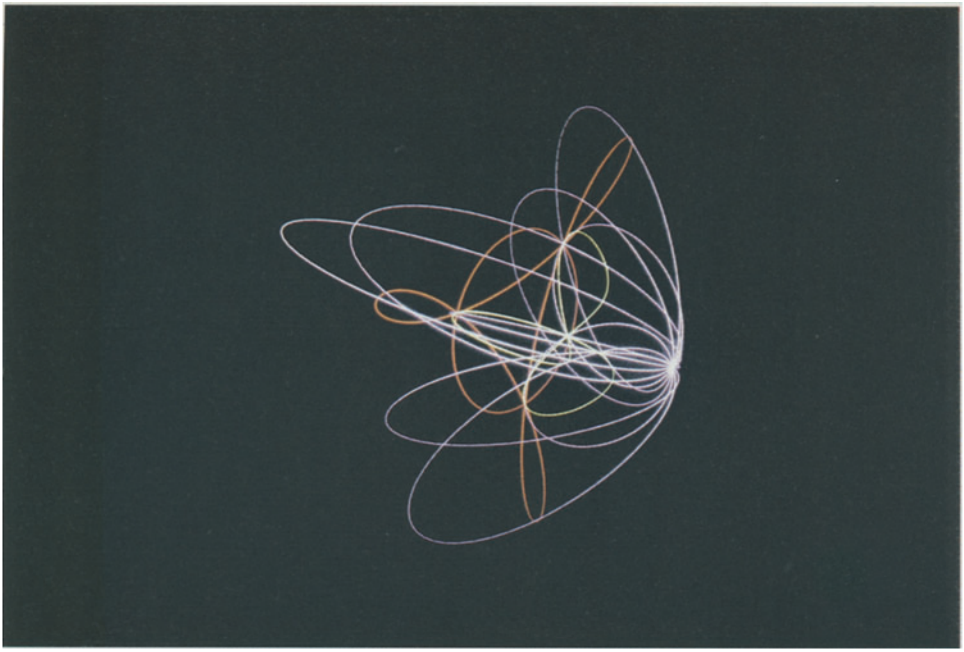




41 Boy surface of degree six

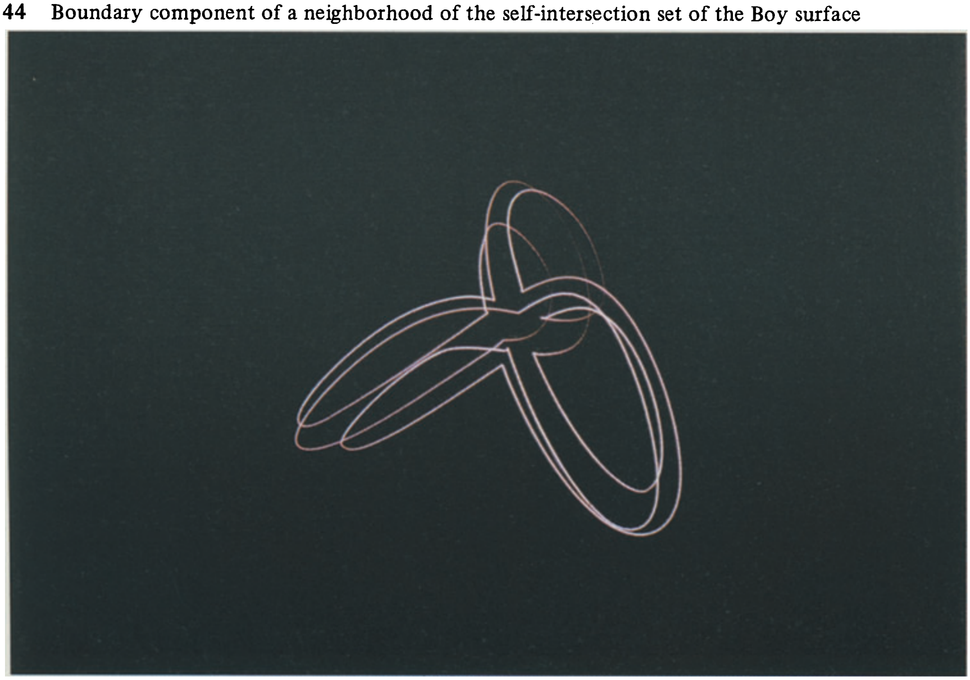
42 Curves of construction of the Boy surface of degree six

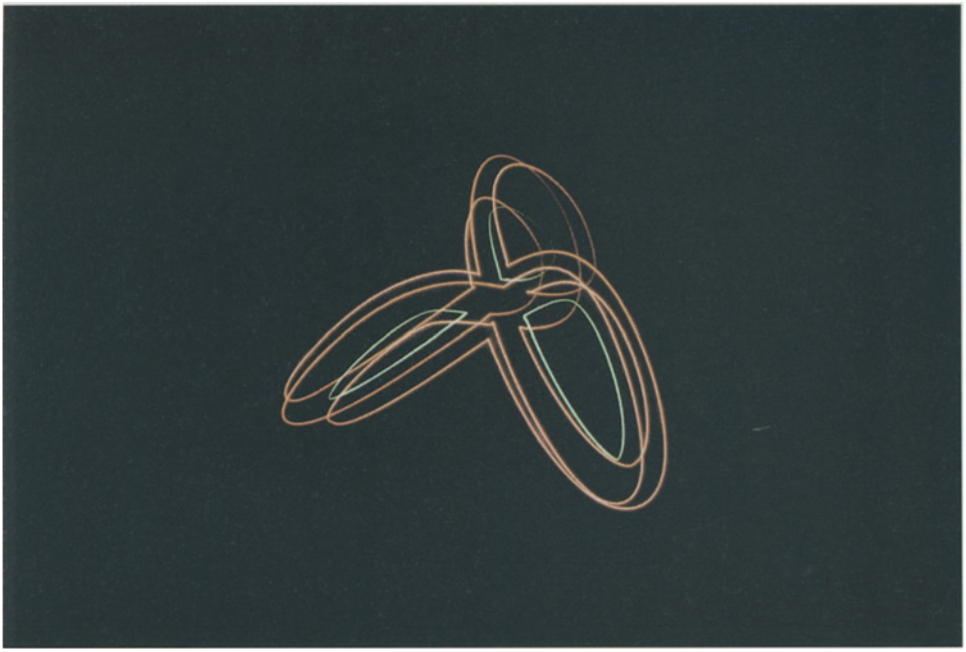




43 Curves of construction of the Boy surface of degree six

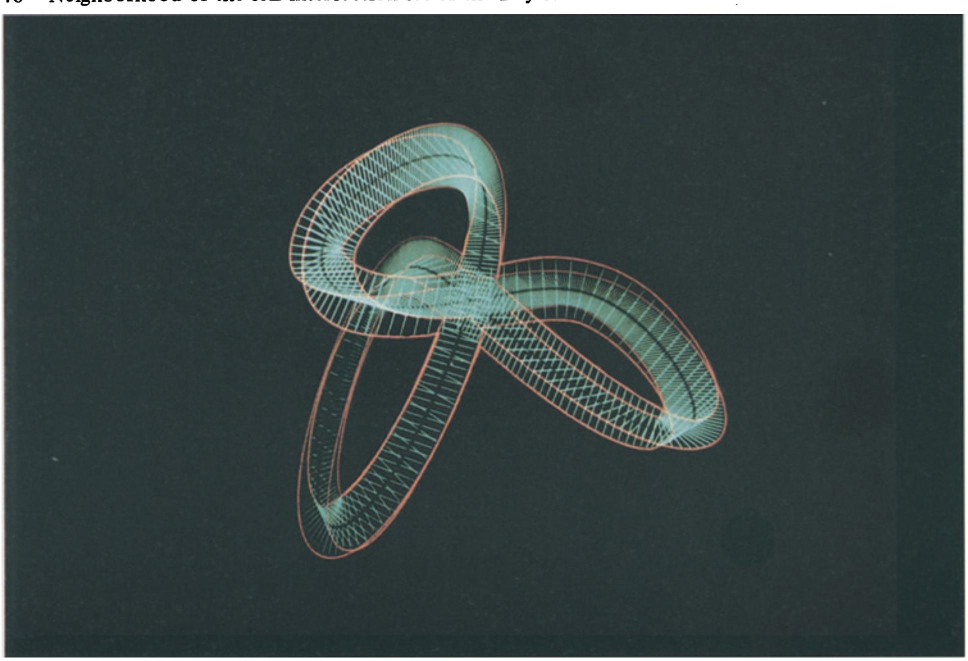
44 Boundary component of a neighborhood of the self-intersection set of the Boy surface

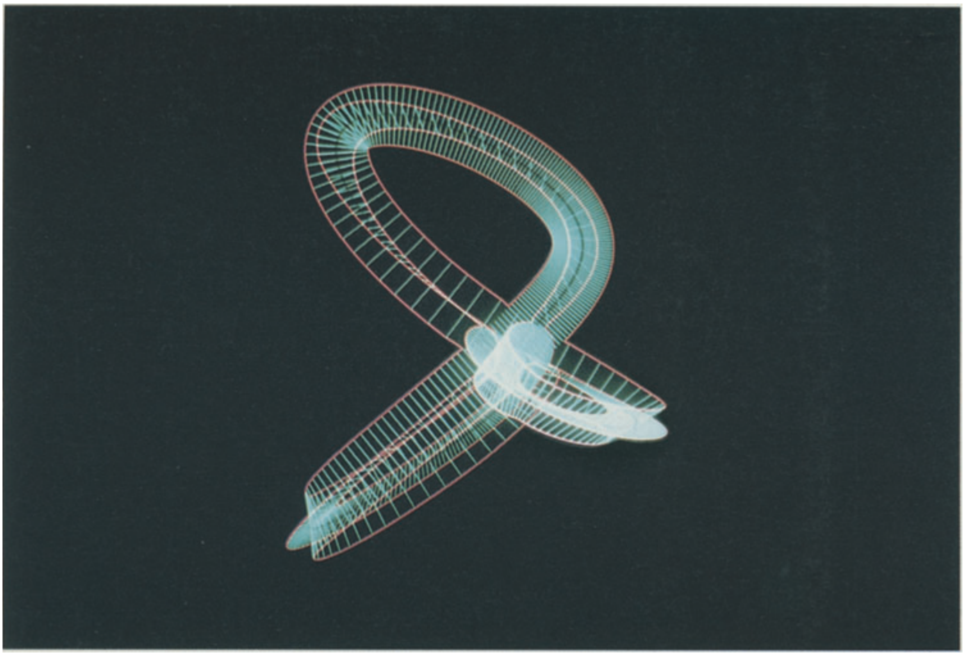




45 Boundary of a neighborhood of the self-intersection set of the Boy surface

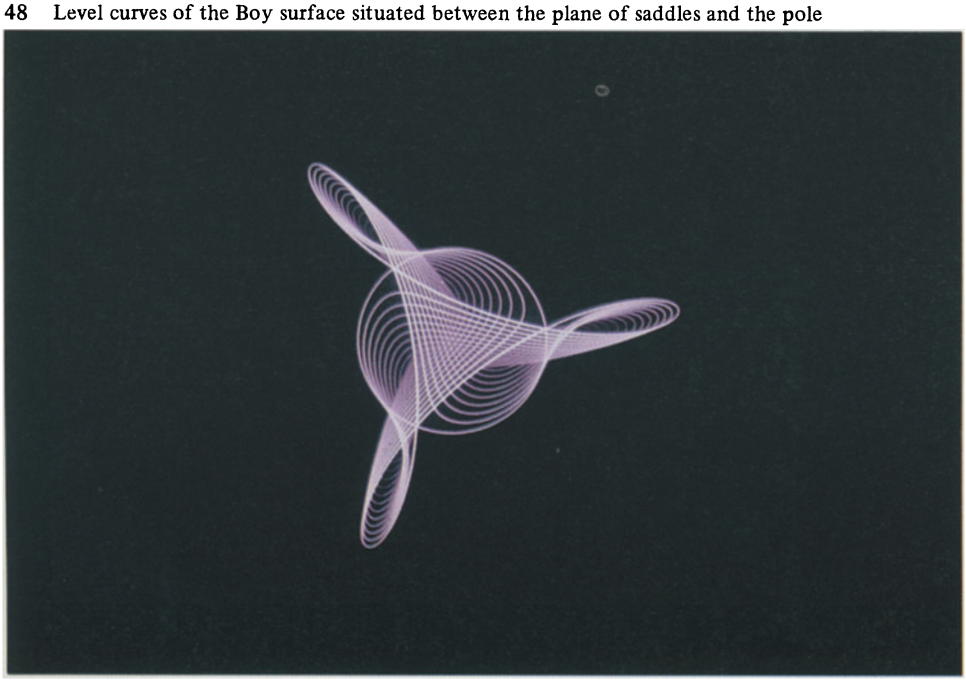
46 Neighborhood of the self-intersection set of the Boy surface

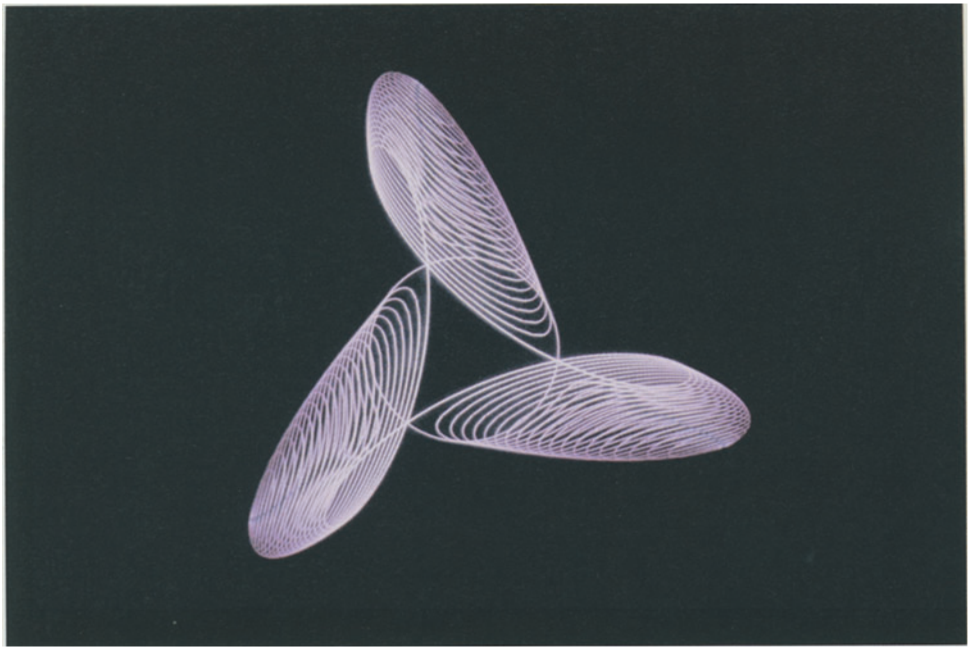




47 Neighborhood of the self-intersection set of the Boy surface

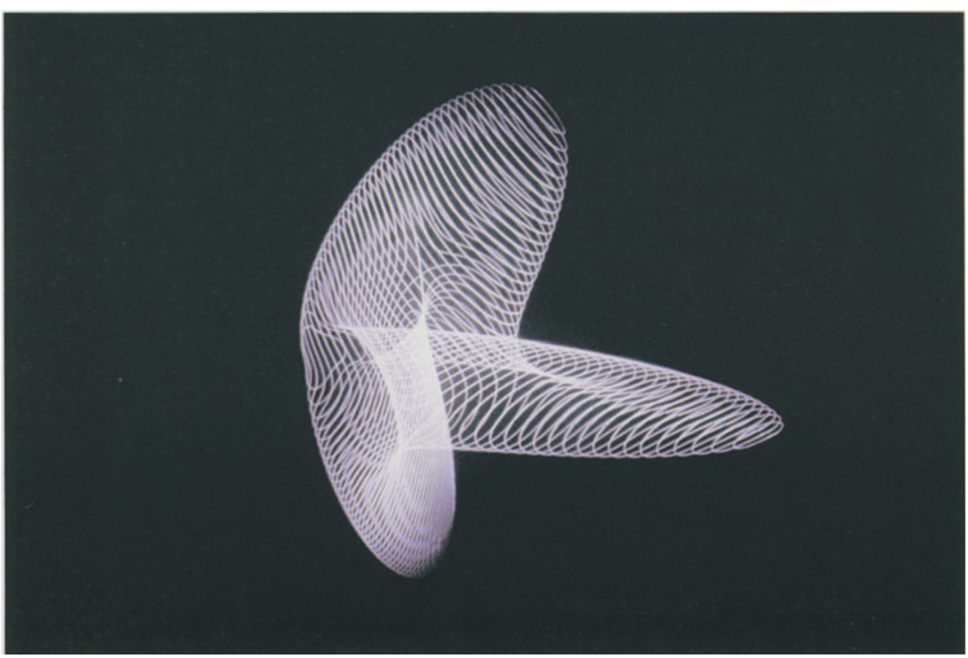
48 Level curves of the Boy surface situated between the plane of saddles and the pole

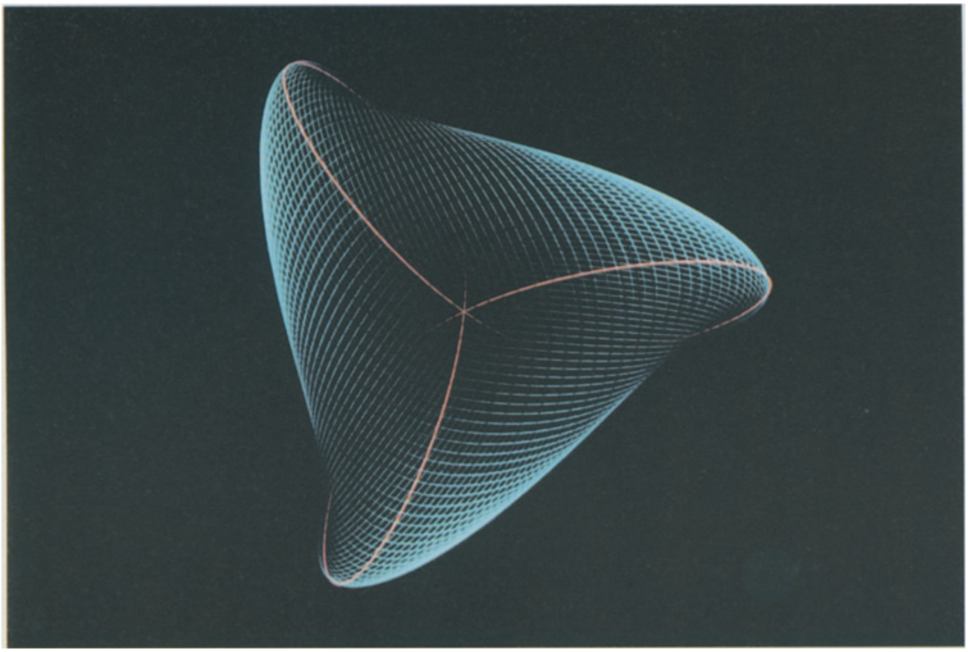




49 Level curves of the Boy surface situated between the plane of saddles and the plane of minima

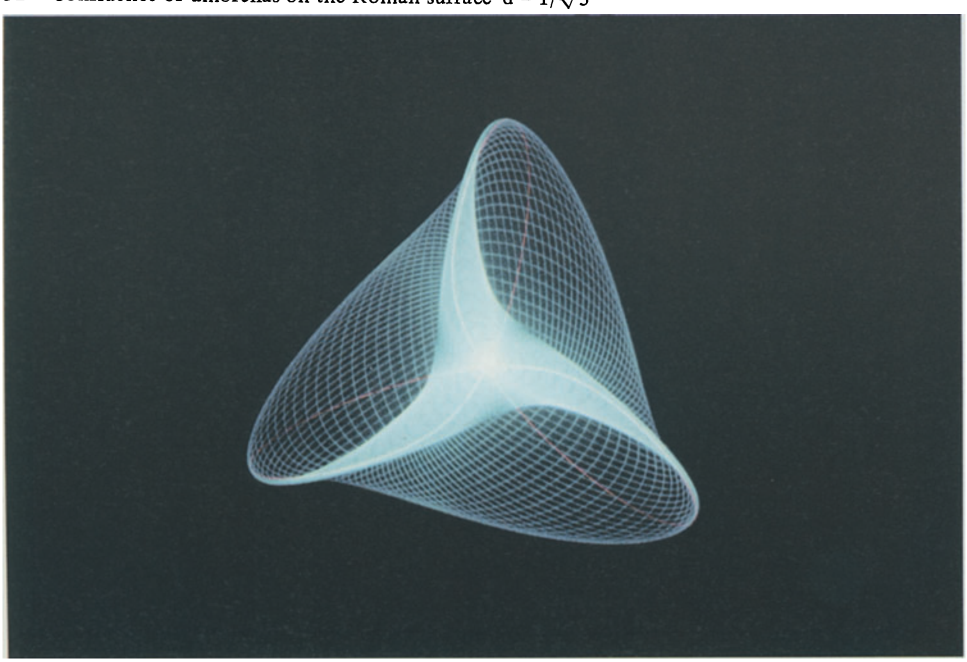
50 Level curves of the Boy surface

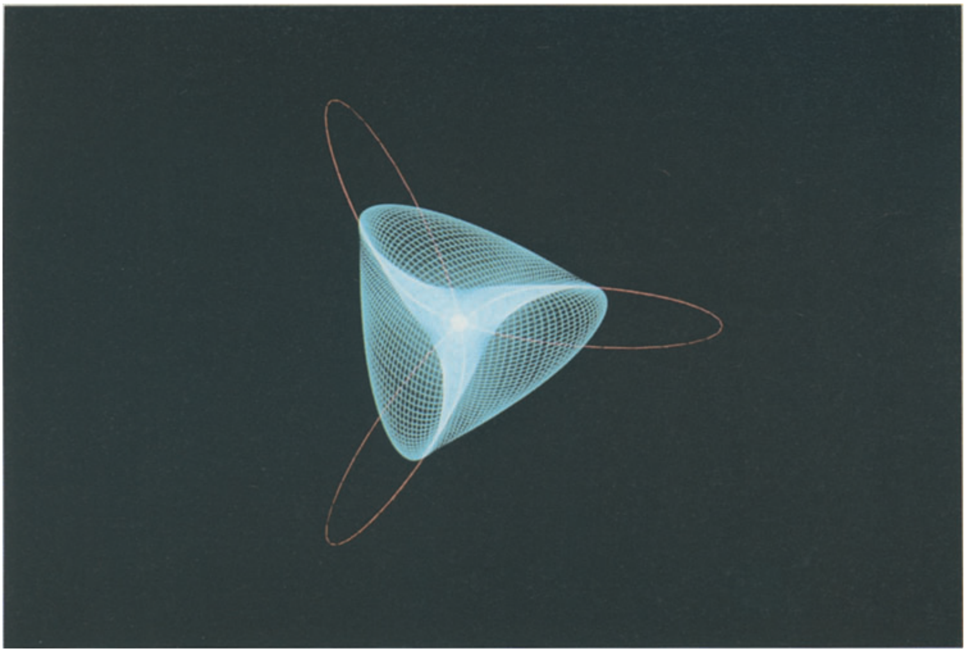




51 Confluence of umbrellas on the Roman surface $d = 1/\sqrt{3}$

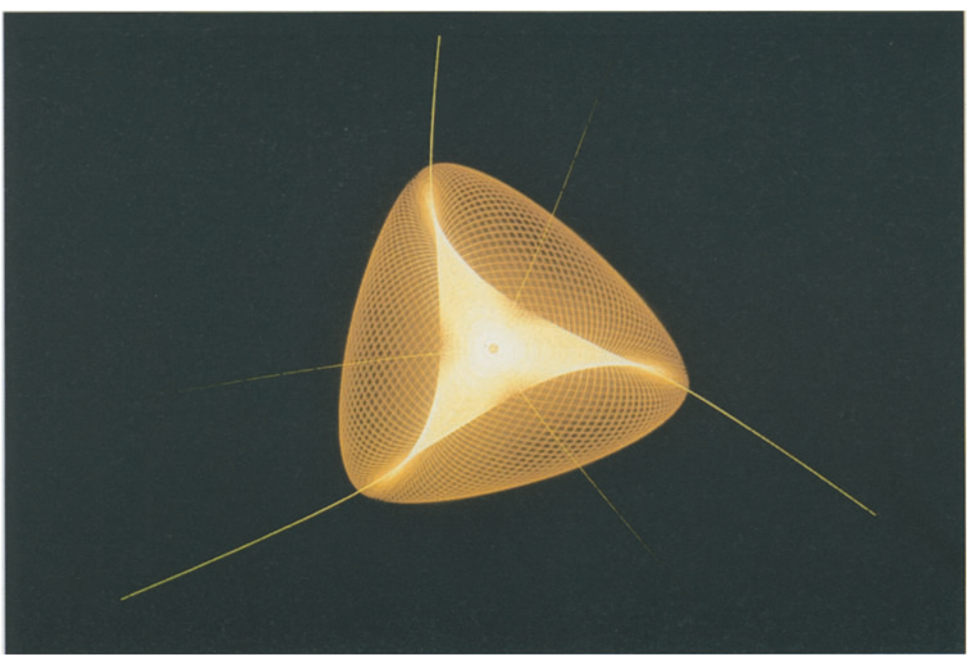
52 Confluence of umbrellas on the Roman surface $d = 1/\sqrt{3}$

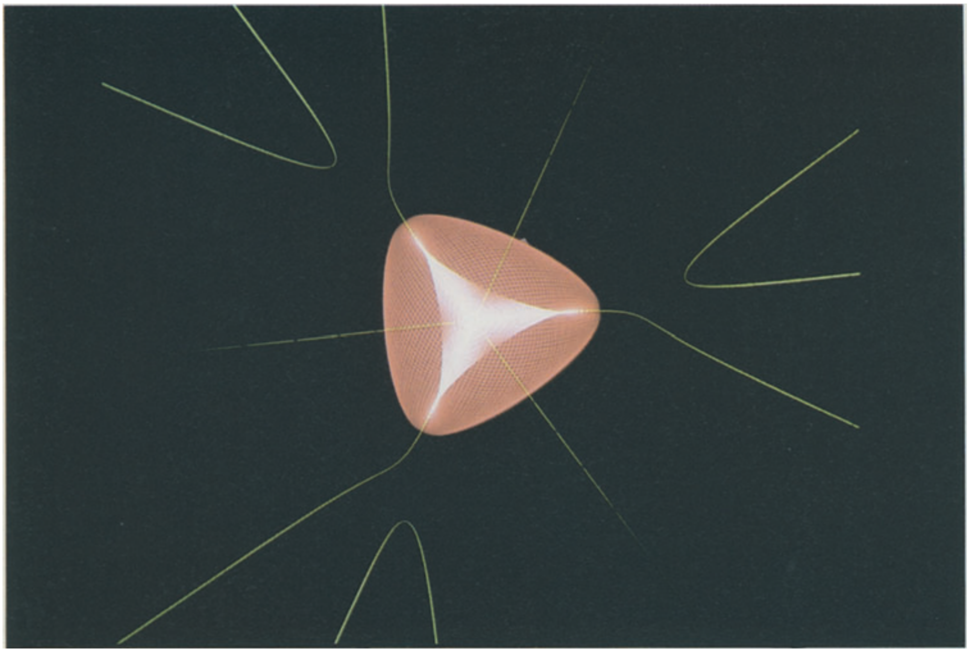




53 Deformation of the Roman surface $d = 0.4$

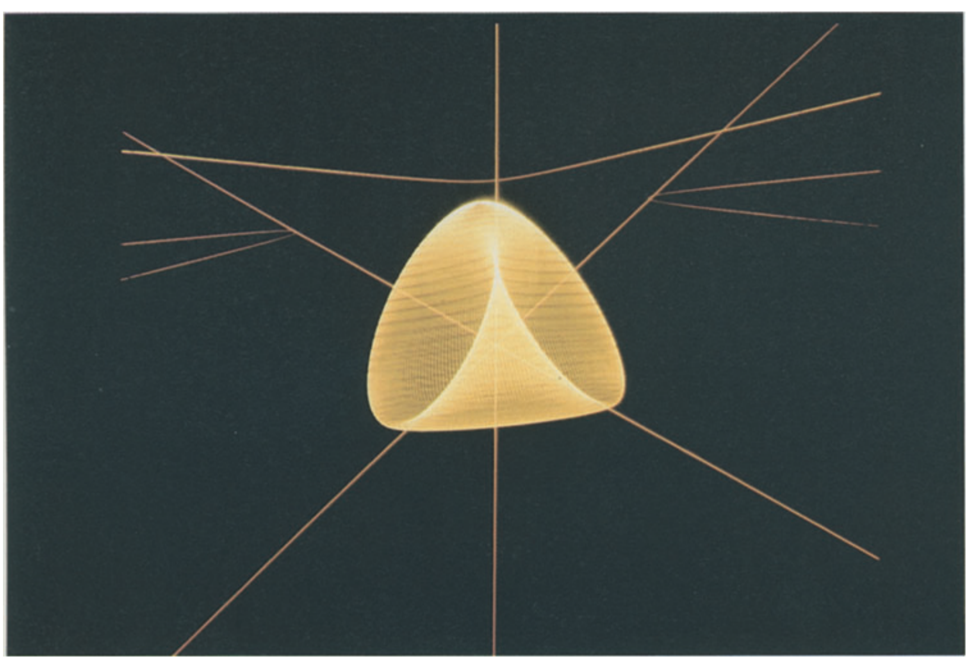
54 Deformation of the Roman surface: the self-intersection curve is tangent to the plane at infinity $d = (\sqrt{2} - 1)^2$

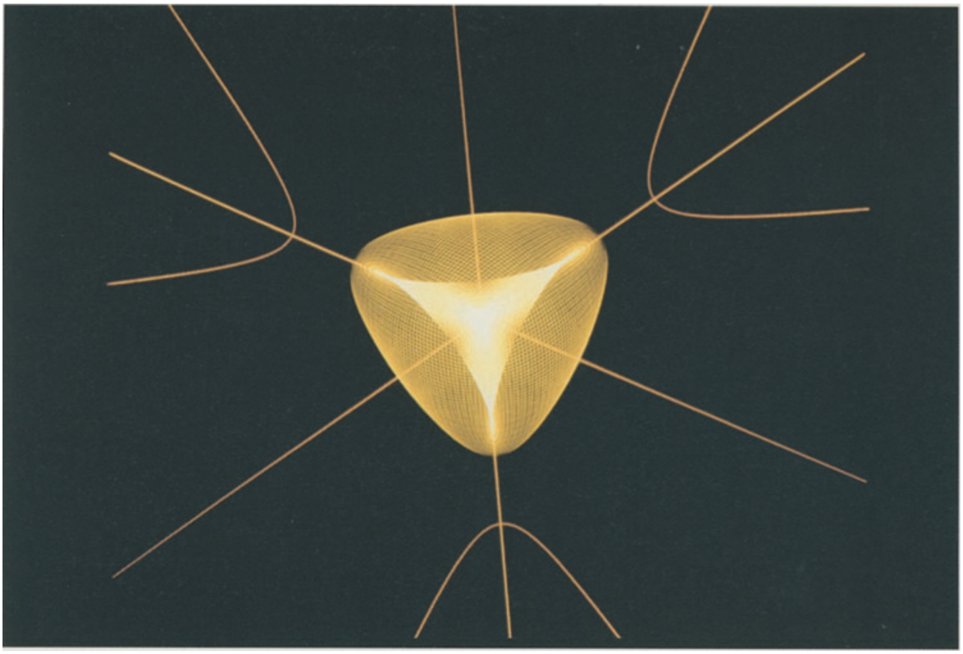




55 Deformation of the Roman surface d = 0.001

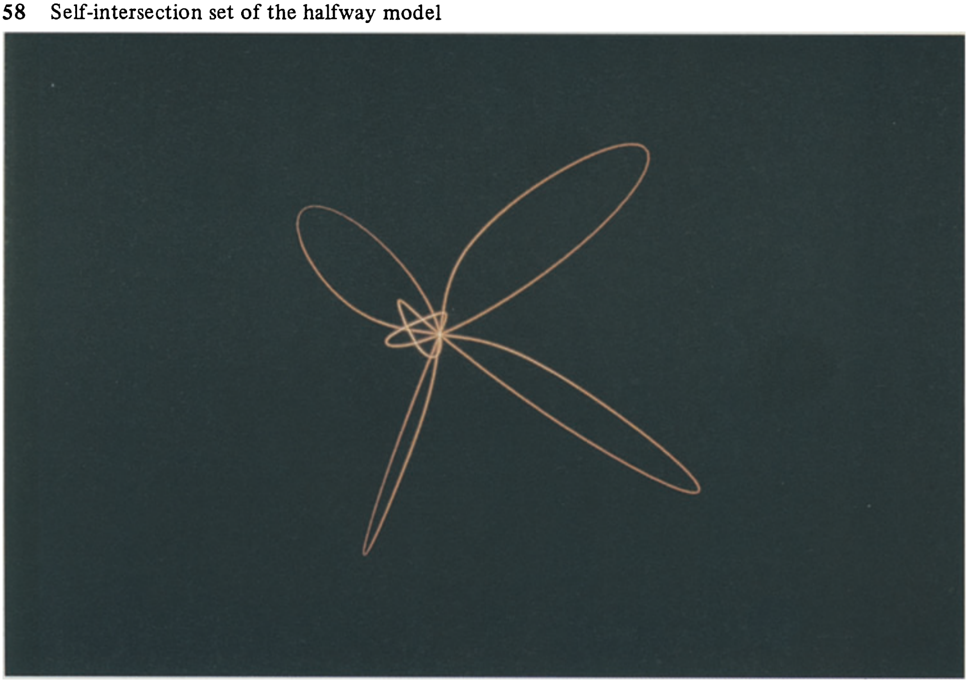
56 Beginning of the deformation of the Roman surface d = 0

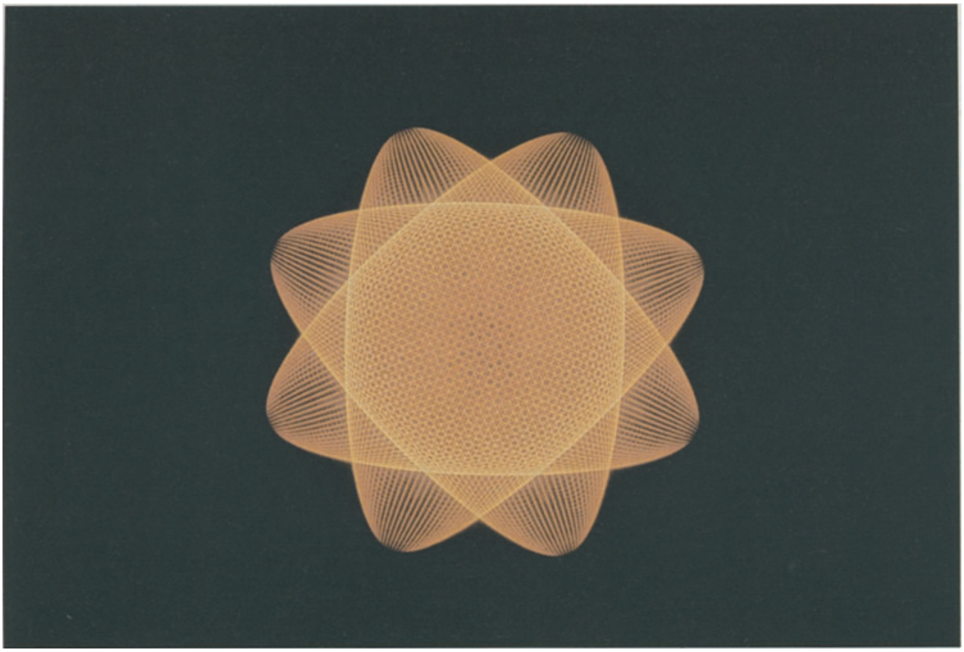




57 Beginning of the deformation of the Roman surface d = 0

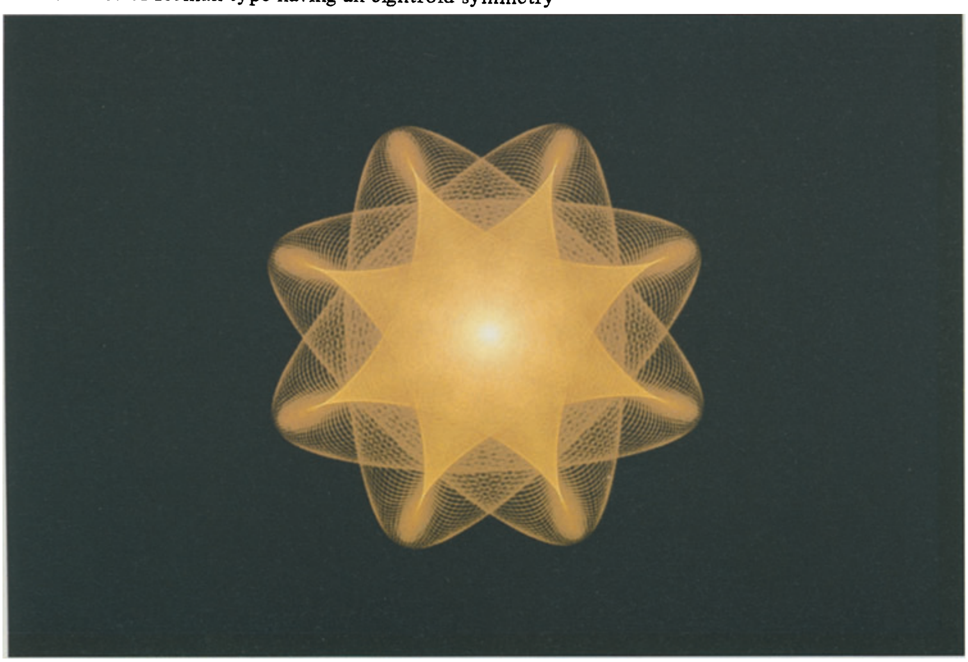
58 Self-intersection set of the halfway model

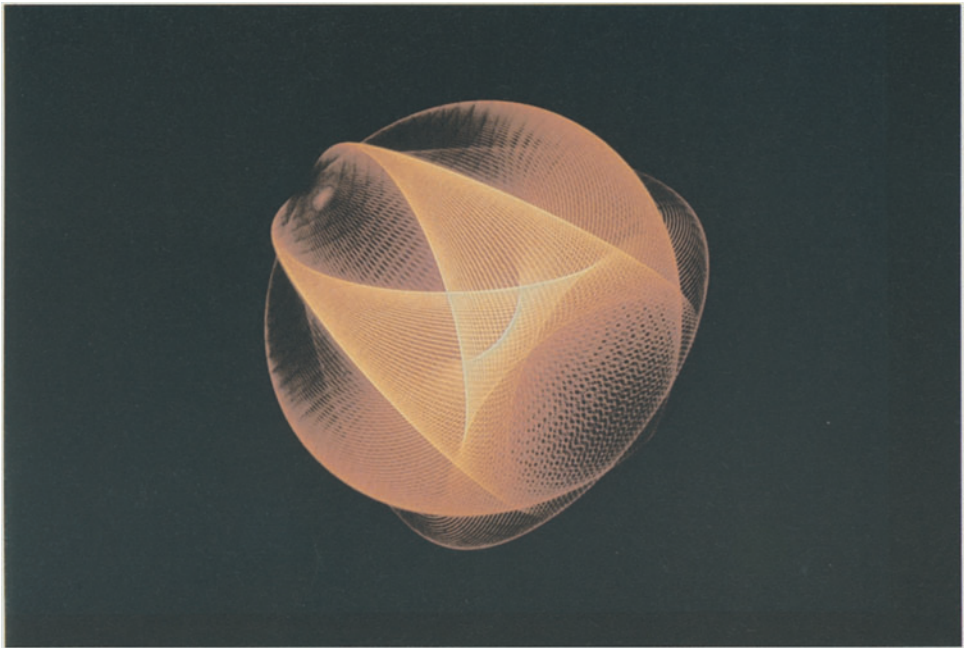




59 Surface of Roman type having an eightfold symmetry

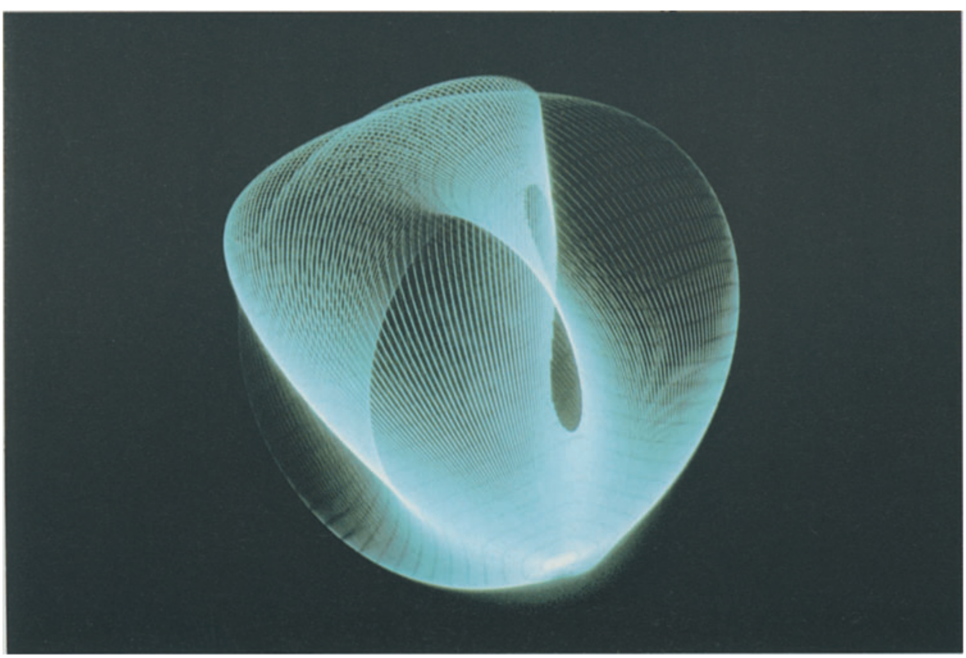
60 Surface of Roman type having an eightfold symmetry

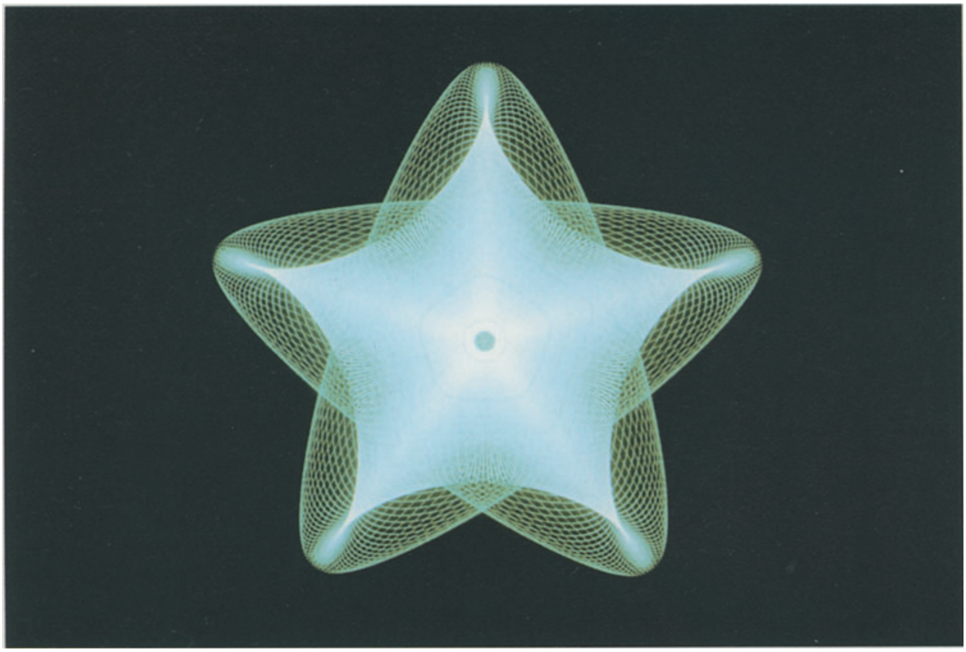




61 Surface of Roman type having an eightfold symmetry

62 Surface of Roman type having a fivefold symmetry with window





63 Surface of Roman type having a fivefold symmetry

64 Immersed projective plane having a fivefold symmetry